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# Conformal Dimensions from Topologically Massive Quantum Field Theory

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**Abstract**

We discuss the evaluation of observables in two-dimensional conformal field theory using the topological membrane description. We show that the spectrum of anomalous dimensions can be obtained perturbatively from the topologically massive quantum field theories by computing radiative corrections to Aharonov-Bohm scattering amplitudes for dynamical charged matter fields. The one-loop corrections in the case of topologically massive Yang-Mills theory are shown to coincide with the scaling dimensions of the induced ordinary and supersymmetric WZNW models. We examine the effects of the dressing of a topologically massive gauge theory by topologically massive gravity and show that the one-loop contributions to the Aharonov-Bohm amplitudes coincide with the leading orders of the KPZ scaling relations for two-dimensional quantum gravity. Some general features of the description of conformal field theories via perturbative techniques in the three-dimensional approach are also discussed.

# 1 Introduction

Conformal field theories in two dimensions [1] have many important applications in theoretical physics, most notably to string theory [2] and critical phenomena in planar systems [3]. In this Paper, the three-dimensional ("membrane") description of conformal field theories will be considered. It is known that two-dimensional conformal field theories can be described in three-dimensional terms by using an amusing connection between them and $(2+1)$-dimensional topological Chern-Simons gauge theories. This connection was first discovered by Witten [4], who found an isomorphism between the finite dimensional space of conformal blocks of the Wess-Zumino-Novikov-Witten (WZNW) model with Kac-Moody symmetry $G_{\rm L}\times G_{\rm R}$ defined on a Riemann surface $\Sigma$, and the finite dimensional Hilbert space of Chern-Simons gauge theory with gauge group $G$ defined on a three-dimensional spacetime manifold $\mathcal{M}=\Sigma\times\mathbb{R}^1$. It was subsequently demonstrated in [5] using path integral techniques that a Chern-Simons theory defined on a three-dimensional manifold $\mathcal{M}$ with two-dimensional boundary $\partial\mathcal{M}$ induces the chiral gauged WZNW model on the world-sheet $\partial\mathcal{M}$ (see [6] for further details). Different coset constructions can be obtained along the same lines using several Chern-Simons theories and the three-dimensional approach is a good starting point to tame the conformal zoo [5].

This connection between apparently distinct quantum field theories provides unexpected and intriguing relations between two- and three-dimensional physics. The connection between a Chern-Simons gauge theory in three dimensions and a conformal field theory in two dimensions was used in [7] to suggest the topological membrane approach to string theory. The modern formulation of string theory is the quantum theory of two-dimensional conformal fields on random surfaces [2]. The basic idea of the topological membrane approach is to fill in the string world sheet and view it as the boundary of a three-manifold. The emphasis of this approach is on the world sheet (rather than target space) properties of the induced string theory.

The induced WZNW model will arise even from the topologically massive gauge theory [8]–[10] defined by the action

$$S_{TMGT}=-\frac{1}{2e^2}\int_{\mathcal{M}}d^3x~{\rm tr}~\sqrt{g}~g^{\mu\lambda}g^{\nu\rho}F_{\mu\nu}F_{\lambda\rho}+S_{CS} \tag{1.1}$$

where the topological action

$$S_{CS}=\int_{\mathcal{M}}\frac{k}{4\pi}~{\rm tr}\left(A\wedge dA+\frac{2}{3}A\wedge A\wedge A\right)=\int_{\mathcal{M}}d^3x~\frac{k}{4\pi}\epsilon^{\mu\nu\lambda}~{\rm tr}\left(A_\mu\partial_\nu A_\lambda+\frac{2}{3}A_\mu A_\nu A_\lambda\right) \tag{1.2}$$

is the integral of the parity-violating Chern-Simons 3-form over the three-dimensional spacetime manifold $\mathcal{M}$ with metric $g_{\mu\nu}$ of Minkowski signature. Here $A=A_\mu(x)dx^\mu=A^a_\mu(x)T^a dx^\mu$

is a gauge connection of the trivial vector bundle over $\mathcal{M}$, where the anti-Hermitian generators $T^a$ of the compact gauge group $G$ are normalized as

$$\mathrm{tr}\ T^a T^b = \frac{1}{2}\delta^{ab} \quad , \tag{1.3}$$

and $F = dA + [A, A]/2$ is the curvature of $A$. Unlike the pure Chern-Simons theory (1.2), the action (1.1) does not define a topological field theory because the Yang-Mills kinetic term $F^2$ for the gauge field depends explicitly on the 3-dimensional metric $g_{\mu\nu}$ and there are propagating degrees of freedom (massive vector bosons) with topological mass

$$M = ke^2/4\pi \tag{1.4}$$

which means that the Hilbert space of this quantum field theory is infinite-dimensional. The presence of induced degrees of freedom on the boundary $\partial\mathcal{M}$ depends solely on the Chern-Simons term in the action and is due to the fact that this term is not invariant under local gauge transformations $A \to A^g = g^{-1}Ag + g^{-1}dg$ with nontrivial gauge functions on the boundary, i.e. $g|_{\partial\mathcal{M}} \neq 0$. When $\pi_3(G) \neq 0$, it is necessary to quantize the Chern-Simons coefficient $k$ so that $k \in \mathbb{Z}$ [10] to ensure that the quantum theory is invariant under large gauge transformations which are not connected to the identity. In the induced two-dimensional WZNW model, the integer $k$ then represents the level of the current algebra generated by the gauge group $G$.

The appearance of a metric tensor in (1.1) is very important because it is the only way to connect gauge degrees of freedom and gravity. Recall that to construct a string theory there must be two ingredients. The first one is a conformal field theory on the string world-sheet and the second is a fluctuating (random) geometry. The latter ingredient can be described by the three-dimensional Einstein gravity action

$$S_E = \kappa \int_{\mathcal{M}} d^3x\ \sqrt{g}\ R \tag{1.5}$$

where $R$ is the curvature of the metric $g_{\mu\nu}$ and $\kappa = 1/2\pi G$ is the Planck mass ($G$ is the gravitational constant). The model (1.5) can be considered as a topological Chern-Simons theory with gauge group $ISO(2,1)$ [11]. It was shown in [12] that in this case an integration over the three-dimensional metric in the bulk interior of $\mathcal{M}$ leads to an integration over the moduli space of two-dimensional complex structures on the boundary $\partial\mathcal{M}$, with the same integration measure that naturally occurs in string theory (the Weil-Petersson measure). The reparametrization-ghost contributions were obtained in [13] using the explicit wavefunctions on moduli space. These wavefunctions were found in [14] in the case of the torus $\Sigma = T^2$.

Besides the Einstein term it is also natural to incorporate the gravitational Chern-Simons term into the gravity action [12, 15, 16]. Indeed, in both the pure gauge and gravitational cases one-loop radiative corrections to the Yang-Mills and Einstein actions induce Chern-Simons

terms in three-dimensions. The quantum geometrodynamics of the topological membrane is described by topologically massive gravity [10] which is the sum of the Einstein and gravitational Chern-Simons terms

$$S_{TMG} = \frac{k'}{8\pi} \int_{\mathcal{M}} d^3x \ \epsilon^{\mu\nu\lambda} \left( R_{\mu\nu\alpha\beta}\omega_\lambda^{\ \alpha\beta} + \frac{2}{3}\omega_{\mu\alpha}^{\ \ \beta}\omega_{\nu\beta}^{\ \ \gamma}\omega_{\lambda\gamma}^{\ \ \alpha} \right) + S_E \tag{1.6}$$

where $\omega_{\mu\alpha}^{\ \ \beta}$ is the Levi-Civita spin connection for $g_{\mu\nu}$. The first term in (1.6) can be regarded as a Chern-Simons action for an $SO(2,1)$ gauge theory with connection $\omega$. The gravity theory (1.6) is no longer topological and there are now propagating graviton degrees of freedom with topological mass

$$\mu = 8\pi\kappa/k' \tag{1.7}$$

In the connection with conformal field theory, the gravitational Chern-Simons coefficient $k'$, which need not be quantized, is proportional to the corresponding two-dimensional central charge $c$. It was shown in [16] that the resulting quantum mechanics on moduli space leads to a holomorphic dependence of the wavefunctions on moduli and it was conjectured, based on the similarity between the $1/k'$ expansion in topologically massive gravity and the $1/c$ expansion in two-dimensional quantum gravity, that the model (1.6) should be related to the gravity sector in string theory, i.e. to Liouville theory [2, 17]. This conjecture was (formally) proved in [18] at the path integral level where it was demonstrated that topologically massive gravity defined on a $(2+1)$-dimensional manifold $\mathcal{M}$ induces two-dimensional quantum gravity, i.e. Liouville theory, on the boundary $\partial\mathcal{M}$. The two-dimensional cosmological constant $\Lambda$, which determines the scale in Liouville theory defined by the action

$$S_L = \int_{\partial\mathcal{M}} d^2z \ \left( \partial_{\bar{z}}\phi\partial_z\phi + QR^{(2)}\phi + \Lambda\, \mathrm{e}^{\alpha\phi} \right) \qquad , \tag{1.8}$$

is equal to the square of the topological graviton mass [19] (see also [20])

$$\Lambda = \mu^2 \tag{1.9}$$

An important issue in the above formal correspondences is to what extent the quantum field theory in the bulk $\mathcal{M}$ describes the quantum characteristics of the induced conformal field theory on $\partial\mathcal{M}$. In this Paper we will discuss how to connect the basic observables of two-dimensional conformal field theory with those of the topologically massive theories using perturbative techniques. More precisely, we shall show that the conformal spin eigenvalues of primary operators in the WZNW and Liouville models arise from a perturbative expansion of scattering amplitudes up to one-loop order in the topologically massive field theories above coupled to charged matter fields. The idea behind this relation is as follows. The observables of two-dimensional conformal field theory can be expressed in terms of the vertex operators

$$V(z,\bar{z}) = V_{\rm L}(z)V_{\rm R}(\bar{z}) \tag{1.10}$$

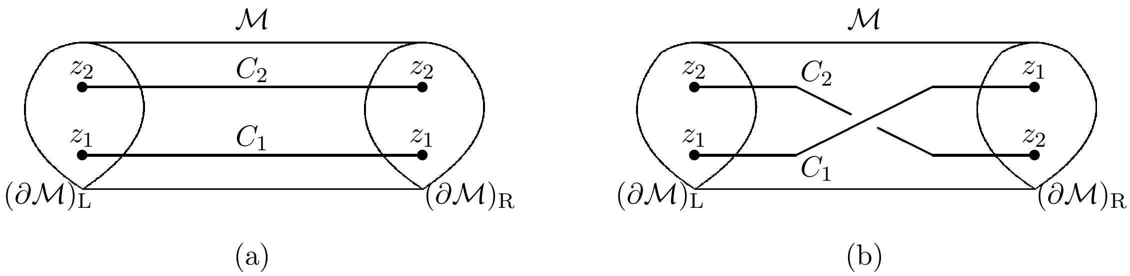



**Figure 1.1:** (a) The filled cylinder $\mathcal{M}$ whose boundaries $(\partial\mathcal{M})_{\rm L,R}$ have punctures $z_i$ corresponding to insertions of the vertex operators $V(z_i,\bar z_i)$. The punctures on the 2 boundaries are joined together by curves $C_i \subset \mathcal{M}$ which represent the propagation of charged particles in the bulk $\mathcal{M}$ whose holonomies are described by the Wilson line operators $W_R[C_i]$. (b) An interchange of punctures on the right-moving world-sheet $(\partial\mathcal{M})_{\rm R}$ is equivalent to the linking of the corresponding Wilson lines in the bulk.

where the subscripts L and R represent the holomorphic and anti-holomorphic sectors of the world-sheet theory. The gauge- and topologically-invariant operators of the topologically massive gauge theory (1.1) are determined by the path-ordered Wilson line operators

$$W_R[C] = \ {\rm tr}_R\ P\exp\left(i\int_C A\right) \tag{1.11}$$

where $C$ is an oriented contour in the 3-manifold $\mathcal{M}$. The Wilson line (1.11) describes the holonomy that arises from adiabatic transport of a charged particle, in a representation $R$ of the gauge group $G$ and with world-line $C$, in the presence of the gauge field $A$. When $\mathcal{M}$ is the filled cylinder depicted in Fig. 1.1 (a), its two boundaries describe the left- and right-moving sectors of the string worldsheet [21]. The Wilson line (1.11) coincides on the left boundary with the holomorphic part $V_{\rm L}(z)$ and on the right boundary with the anti-holomorphic part $V_{\rm R}(\bar z)$ of the vertex operator (1.10). Thus the correlation functions in the two-dimensional conformal field theory are related to correlation functions of the topologically massive gauge theory (1.1). The interchange of two points $z_1$ and $z_2$ on the worldsheet is equivalent to the linking of the corresponding Wilson lines (see Fig. 1.1 (b)).

In the simplest case of two-dimensional free bosons, i.e. a $c=1$ conformal field theory, the chiral vertex operator is

$$V_q(z) = \ {\rm e}^{iq\phi(z)} \tag{1.12}$$

which can be obtained from the Wilson line in the corresponding three-dimensional $U(1)$ theory (the representation $R$ in this case is parametrized by an abelian charge $q$)

$$W_q[C] = \exp\left(iq\int_C A\right) \tag{1.13}$$

with the conformal field $\phi$ as usual identified with the surviving degree of freedom of the gauge field $A$ on $\partial\mathcal{M}$. It is known that in this simple abelian case the short-distance operator

product expansion gives

$$\langle V_q(z_1)V_q(z_2)\rangle \sim (z_1 - z_2)^{-q^2\Delta} \tag{1.14}$$

for $z_1 \sim z_2$, where $\Delta$ is the scaling dimension of the corresponding chiral vertex operators. Thus the interchange operation depicted in Fig. 1.1 (b) induces a phase factor $\mathrm{e}^{2\pi i q^2\Delta}$ in the vacuum expectation value $\langle V_q(z_1)V_q(z_2)\rangle$. A similar phase factor appears in the topologically massive gauge theory correlator $\langle W_q[C_1]W_q[C_2]\rangle$ in the bulk $\mathcal{M}$ due to the linking of the two corresponding Wilson lines (when the minimal distance between the 2 charged particles is much larger than the Compton length of the topologically massive vector boson), where $\Delta$ is the transmuted spin of the charged particles due to their interaction with the Chern-Simons gauge field. This latter phase factor is manifested in the Aharonov-Bohm part of the charged particle-particle scattering amplitudes in the gauge theory [22]. Based on the above correspondences, we expect the phase $\Delta$ to be the same in the two models. In the non-abelian case these heuristic arguments are not precise, but we still expect the same sort of relations between observables of the two- and three-dimensional quantum field theories. A gravitational dressing of the above theories by the topologically massive or Liouville gravity models would correspond to graviton exchanges between the Wilson lines depicted in Fig. 1.1. In this Paper we shall demonstrate the equivalences between the anomalous spins in the two- and three-dimensional cases.

In the following, we shall compute one-loop radiative corrections to the Aharonov-Bohm amplitudes for charged particle scattering in the topologically massive gauge and gravity theories. We shall show that the anomalous dimensions for the WZNW model [23] appear in this way from the non-abelian topologically massive gauge theory (1.1) coupled to dynamical charged scalar fields. For supersymmetric topologically massive Yang-Mills theory, we get the anomalous dimensions for the $N = 1$ supersymmetric WZNW model [24]. Then, we shall study the effects of gravitational dressing on the Aharonov-Bohm amplitudes and demonstrate that the one-graviton exchange corrections to Aharonov-Bohm scattering coincide with the leading correction to the bare conformal dimension predicted by the celebrated Knizhnik-Polyakov-Zamolodchikov (KPZ) formula [17]. In this way, the topological membrane approach suggests a geometric origin for the KPZ scaling relations.

The organization of this Paper is as follows. In Section 2 we briefly discuss the structure of the Aharonov-Bohm amplitudes in Chern-Simons theory and how they can be used to perturbatively determine anomalous dimensions of the associated conformal field theories. In Section 3 we go through the standard calculation of the one-loop renormalization of the Chern-Simons coefficient $k$ in the topologically massive gauge theory (1.1) coupled to charged matter fields and show that the corresponding Aharonov-Bohm amplitude leads to the anticipated spin. There we find that the correct dimension is largely determined by using an appropriate Slavnov-Taylor identity for the gauge field theory. In Section 4 we then include the coupling of

the topologically massive gauge theory to topologically massive gravity. There we demonstrate that the standard set of Ward-Takahashi identities in the presence of the gravitational dressing still apply and we use them to evaluate a large set of Feynman diagrams contributing to the full scattering amplitude. The remaining graviton exchange diagrams are then evaluated explicitly and we discuss the structure of them and how they reproduce anticipated features of the conformal field theory. Section 5 contains some concluding remarks and two Appendices at the end of the Paper are devoted to technical details of the evaluation of some of the more complicated Feynman graphs.

## 2 Aharonov-Bohm Amplitudes and Anomalous Spin in Conformal Field Theory

In this Section we shall describe the general form of the Aharonov-Bohm amplitude in Chern-Simons theory and how it can be used for a perturbative evaluation of conformal dimensions in the topologically massive field theories. We shall assume here and in the following that the Chern-Simons coefficients $k$ and $k'$ are large enough so that the perturbative expansions of the respective quantum field theories make sense. We consider the minimal coupling of the Chern-Simons gauge field $A$ to a conserved current $J^\mu = J^\mu_a R^a$, with $R(G)$ some irreducible unitary representation of $G$ whose generators are normalized as

$$\mathrm{tr}\ R^a R^b = T_R(G)\delta^{ab} \qquad , \tag{2.1}$$

by adding the term

$$S_J = \int d^3x\ 2\ \mathrm{tr}\ J^\mu A^a_\mu R^a \tag{2.2}$$

to the Chern-Simons action (1.2). From (1.2) it follows that the momentum space bare gluon propagator in the transverse Landau gauge (i.e. see the next Section) is

$$\mathcal{G}^{ab}_{\mu\nu}(p) = \left\langle A^a_\mu(p) A^b_\nu(-p)\right\rangle_A = -\frac{4\pi}{k}\delta^{ab}\frac{\epsilon_{\mu\nu\lambda}p^\lambda}{p^2} \tag{2.3}$$

where the expectation value is taken in the non-interacting part of the gauge theory (1.2). The invariant amplitude for the scattering of two charged particles of initial momenta $p_1$ and $p_2$ represented by the current $J$ is then of the form

$$\mathcal{A}(p_1,p_2;q) \equiv i\ \mathrm{tr}\ J^\mu(2p_1-q)\mathcal{G}_{\mu\nu}(q)J^\nu(2p_2+q) = -\frac{16\pi i}{k}\dim(G)T_R(G)f_G(k)\frac{\epsilon_{\mu\nu\lambda}p_1^\mu p_2^\nu q^\lambda}{q^2} \tag{2.4}$$

where $q$ is the momentum transfer and we have used the transversality of the gauge propagator $p^\mu\mathcal{G}_{\mu\nu}(p) = 0$ and current conservation $p_\mu J^\mu(p) = 0$. Here $f_G(k) = \sum_{n\geq 0} f_n/k^n$ is some

function whose coefficients $f_n$ depend only on invariants of the gauge group $G$ and which can be computed perturbatively order by order in the Chern-Simons coupling constant $1/k$.

In the center of momentum frame the amplitude (2.4) becomes

$$\mathcal{A}(p_1,p_2;q)=\frac{16\pi i}{k}\dim(G)T_R(G)f_G(k)\frac{\vec{p}\times\vec{q}}{\vec{q}^2} \tag{2.5}$$

This amplitude is none other than the Aharonov-Bohm amplitude for the scattering of a charge of strength $\sqrt{T_R(G)f_G(k)}$ off of a flux of strength $(4\pi/k)\dim(G)\sqrt{T_R(G)f_G(k)}$. This is the standard argument for the appearence of induced fractional spin and statistics perturbatively in a Chern-Simons gauge theory and it leads to the spin factor

$$\Delta_G(k)=\frac{T_R(G)}{k}f_G(k) \tag{2.6}$$

which measures the anomalous change of phase in the Aharonov-Bohm wavefunction under adiabatical rotation of one charged particle about another in the gauge theory (1.2) [22]. Thus computing the function $f_G(k)$ order by order in perturbation theory allows us to check the standard formulas for anomalous spin in the Chern-Simons field theories.

There are two well-known conformal dimension formulas that can be explored perturbatively in this way. It was shown by Knizhnik and Zamolodchikov [23] that the conformal weights of the primary operators in a current algebra based on a compact semi-simple Lie group $G$ are given by

$$\Delta_{KZ}=\frac{T_R(G)}{k+C_2(G)} \tag{2.7}$$

where $k\in\mathbb{Z}$ is the level of the WZNW model, $R(G)$ is an irreducible unitary representation of $G$ carried by the primary conformal fields, and $C_2(G)$ is the quadratic Casimir operator in the adjoint representation of $G$, i.e.

$$C_2(G)\delta^{ab}=f^{acd}f^{bcd} \tag{2.8}$$

with $f^{abc}$ the antisymmetric structure constants of the Lie group $G$,

$$[T^a,T^b]=f^{abc}T^c \tag{2.9}$$

The perturbative expansion (2.6) can be therefore compared with the large-$k$ expansion of the weights (2.7) for the WZNW model which is

$$\Delta_{KZ}=\frac{T_R(G)}{k}\left[1-\frac{C_2(G)}{k}+\left(\frac{C_2(G)}{k}\right)^2-\dots\right] \tag{2.10}$$

In the case of the 2-dimensional Liouville theory, it is known from the KPZ random geometry approach [17] that the bare conformal dimensions $\Delta_0$ of primary fields undergo a simple quadratic transformation

$$\Delta-\Delta_0=\frac{\Delta(1-\Delta)}{\tilde{k}+2} \tag{2.11}$$

as a result of the gravitational dressing. Here $\tilde{k}$ is the central charge of the $SL(2,\mathbb{R})$ current algebra. It is related to the gravitational Chern-Simons coefficient in (1.6) by $\tilde{k} = -k' - 4$ (the coefficient of the Liouville kinetic term in (1.8) is $k'$ after an appropriate rescaling of the conformal field $\phi$). The solution of (2.11) for the gravitationally-dressed weight $\Delta$ with the branch satisfying the "natural" boundary condition $\Delta(\Delta_0 = 0) = 0$ is

$$\Delta = -\frac{(k'+1)}{2}\left(1 - \sqrt{1 + \frac{4\Delta_0(k'+2)}{(k'+1)^2}}\right) \tag{2.12}$$

whose expansion for large $k'$ is

$$\Delta = \Delta_0 + \frac{\Delta_0 - \Delta_0^2}{k'} + \frac{2\Delta_0^3 - \Delta_0 - \Delta_0^2}{k'^2} + \ldots \tag{2.13}$$

Thus, coupling the gravitational theory to charged matter of conformal weight $\Delta_0$, one can also compute the gravitational corrections to the Aharonov-Bohm amplitudes above perturbatively in $1/k'$ and check it with (2.13).

# 3 Conformal Weights in Topologically Massive Gauge Theory

In this Section we shall demonstrate, as a preliminary discussion, the appearence of the Knizhnik-Zamolodchikov formula perturbatively up to one-loop order in Chern-Simons gauge theory. This calculation is essentially the well-known computation of the one-loop corrections to the Chern-Simons coefficient $k$ and we shall therefore only highlight the details of the calculation for the sake of comparison with the corresponding calculations that we will present later on for the gravitational contributions. We assume here that $\mathcal{M}$ is flat Minkowski spacetime with metric $g = \operatorname{diag}(1,-1,-1)$. In the next Section we will remove this constraint and consider the effects of gravitational dressing. In the previous Section we saw that scattering amplitudes in the Chern-Simons gauge theory correspond to the familiar ones that appear from the Aharonov-Bohm effect. However, the Chern-Simons gauge theory (1.2) is only strictly-renormalizable and the quantum field theory should be properly defined with an ultraviolet cutoff. We therefore add a non-topological parity-symmetric Yang-Mills kinetic term to the action and consider the topologically massive gauge theory (1.1) instead. The dimensionful parameter $e^2$ acts as an ultraviolet cutoff, and the effective coupling constant of the perturbation expansion is $e^2/M \sim 1/k$. The appearence of a massive, propagating gluon in the model removes the infrared divergences and makes the quantum field theory with action (1.1) a power-counting super-renormalizable field theory with finite, computable

physical parameters [25]–[27]. The Aharonov-Bohm amplitudes (2.4) can then be obtained from the imaginary, parity-odd parts of the amplitudes in the topologically massive gauge theory in the infrared limit $e^2 \to \infty$ when the Yang-Mills action in (1.1) becomes irrelevant and the resulting topological field theory induces the WZNW model on $\partial\mathcal{M}$. This limit, wherein the only effect of the gluon field is to transmute the spin and statistics of external charged particles, is often refered to as the 'anyon limit'[1].

We use the standard Faddeev-Popov gauge-fixing procedure to fix a covariant gauge by introducing anticommuting ghost fields $\eta^a, \bar\eta^a$ in the adjoint representation of $G$ and adding the action

$$S_g = \int d^3x \, \left( \frac{1}{\xi} \, \mathrm{tr}(\partial_\mu A^\mu)^2 + (\partial^\mu \bar\eta^a)(\partial_\mu \eta^a + i f^{abc} A^b_\mu \eta^c) \right) \tag{3.1}$$

to (1.1), where $\xi$ is the covariant gauge fixing parameter and the ghost field action in (3.1) is minimally coupled to the gauge field $A$. It is then straightforward to write down the Feynman rules for this gauge theory. The bare gluon propagator (2.3) is replaced by

$$G^{ab}_{\mu\nu}(p) = \delta^{ab} \left\{ -ie^2 \left( \frac{p^2 g^\perp_{\mu\nu}(p) + iM\epsilon_{\mu\nu\lambda}p^\lambda}{p^2(p^2 - M^2)} \right) + \xi \frac{p_\mu p_\nu}{(p^2)^2} \right\} \tag{3.2}$$

where

$$g^\perp_{\mu\nu}(p) = g_{\mu\nu} - p_\mu p_\nu / p^2 \tag{3.3}$$

is the symmetric transverse projection operator in momentum space with $p^\mu g^\perp_{\mu\nu}(p) = 0$. From (3.1) it follows that the bare ghost propagator in momentum space is

$$\tilde G^{ab}(p) = \langle \bar\eta^a(p) \eta^b(-p) \rangle_\eta = i\delta^{ab}/p^2 \tag{3.4}$$

The bare ghost-ghost-gluon vertex function is

$$\tilde\Gamma^{abc}_\mu(p,q;r) = -if^{abc} p_\mu \tag{3.5}$$

where $r = p + q$, the bare 3-point gluon vertex function is

$$\Gamma^{abc}_{\mu\nu\lambda}(p,q,r) = \frac{i}{e^2} f^{abc} \left( iM\epsilon_{\mu\nu\lambda} + (p-q)_\lambda g_{\mu\nu} + (q-r)_\mu g_{\nu\lambda} + (r-p)_\nu g_{\lambda\mu} \right) \tag{3.6}$$

where $p + q + r = 0$, and the bare 4-point gluon vertex is given by

$$\begin{aligned} \Gamma^{abcd}_{\mu\nu\sigma\lambda}(p,q,r,s) = \frac{i}{e^2} \Big[ & f^{eab} f^{ecd} \left( g_{\mu\lambda} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\lambda} \right) + f^{eac} f^{edb} \left( g_{\mu\sigma} g_{\lambda\nu} - g_{\mu\nu} g_{\lambda\sigma} \right) \\ & + f^{ead} f^{ebc} \left( g_{\mu\nu} g_{\sigma\lambda} - g_{\mu\lambda} g_{\sigma\nu} \right) \Big] \end{aligned} \tag{3.7}$$

with $p + q + r + s = 0$.

[1]Note that in the anyon limit the system is essentially non-relativistic. In fact, the Aharonov-Bohm scattering can be investigated using the formalism of non-relativistic quantum field theory. There, however, the role of the ultraviolet regulator (played in the present relativistic formalism by the Yang-Mills kinetic term) can be played by a quartic contact interaction among the matter fields [28].

## 3.1 Coupling to Scalar Fields

We first consider the minimal coupling of the gauge theory defined above to dynamical charged massive scalar fields with action

$$S_m = \int d^3x \, \left( \left| \partial_\mu \phi_A - i R^a_{AB} A^a_\mu \phi_B \right|^2 - m^2 |\phi_A|^2 \right) \tag{3.8}$$

where $R(G)$ is the representation of $G$ carried by the charged mesons and $m$ is their mass. Then the bare scalar propagator is

$$S_{BA}(p) = \langle \phi^*_A(p) \phi_B(-p) \rangle_\phi = \frac{i}{p^2 - m^2} \delta_{AB} \tag{3.9}$$

and the bare meson-meson-gluon vertex function is $-iR^a_{AB}(p+p')_\mu$ (with $q = p - p'$ the gluon momentum) while the meson-meson-gluon-gluon vertex is $ig_{\mu\nu}\{R^a, R^b\}_{AB}$. To evaluate the one-loop order conformal weight $\Delta^{(1)}$ in this topologically massive gauge theory, we consider the Feynman graphs corresponding to the charged meson-meson scattering amplitude up to one-loop order. We calculate all amplitudes in the following in the transverse Landau gauge ($\xi = 0$ in (3.2)) and we shall always assume that the external scalar particles are on-shell. The tree-level amplitude is easily calculated to be (Fig. 3.1)

$$\begin{aligned} \mathcal{T}(p_1, p_2; q) &= -i \ \mathrm{tr} \ R^a (2p_1 - q)^\mu G^{ab}_{\mu\nu}(q) R^b (2p_2 + q)^\nu \\ &= -\frac{4e^2 \dim(G) T_R(G)}{q^2(q^2 - M^2)} \left\{ \left[ q^2 (p_1 \cdot p_2) - (p_1 \cdot q)(p_2 \cdot q) \right] + iM \epsilon_{\mu\nu\lambda} p_1^\mu p_2^\nu q^\lambda \right\} \end{aligned} \tag{3.10}$$

which in the anyon limit is

$$\lim_{M \to \infty} \mathcal{T}(p_1, p_2; q) = \frac{16\pi i \dim(G) T_R(G)}{k} \frac{\epsilon_{\mu\nu\lambda} p_1^\mu p_2^\nu q^\lambda}{q^2} \tag{3.11}$$

This identifies the tree-level conformal weight

$$\Delta^{(0)} = T_R(G)/k \tag{3.12}$$

which agrees with the leading order term in the expansion (2.10) in $1/k$.

Next we evaluate the corrections to the internal gluon line in Fig. 3.1 from the gluon, ghost and scalar loops, and the gluon and scalar tadpoles (see Fig. 3.2). These yield the amplitude

$$\mathcal{G}(p_1, p_2; q) = -i \ \mathrm{tr} \ R^a (2p_1 - q)^\mu R^b (2p_2 + q)^\nu G^{ac}_{\mu\lambda}(q) \Pi^{\lambda\rho}_{cd}(q) G^{db}_{\rho\nu}(q) \tag{3.13}$$

where $\Pi^{ab}_{\mu\nu}(q)$ is the total 1-loop vacuum polarization tensor which renormalizes the bare gluon propagator as $G^{ab}_{\mu\nu}(q)^{-1} \to G^{ab}_{\mu\nu}(q)^{-1} + \Pi^{ab}_{\mu\nu}(q)$. Since $p^\mu p^\nu G^{ab}_{\mu\nu}(q) = \xi \delta^{ab}$, by gauge invariance

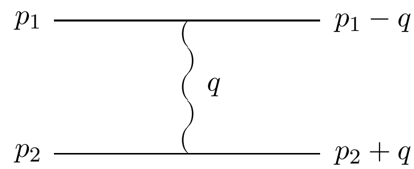



**Figure 3.1:** Tree-level amplitude in topologically massive Yang-Mills theory. The conventions to be used for all Feynman diagrams in the following are that $p_1$, $p_2$ denote the incoming matter momenta, and $q$ is the momentum transfer. Straight lines denote matter fields while wavy lines depict gluons.

it follows that the longitudinal part is not renormalized and the gluon self energy is transverse in $q$,

$$\Pi^{ab}_{\mu\nu}(q) = \delta^{ab}\left\{\frac{q^2 g^{\perp}_{\mu\nu}(q)}{M}\Pi_e(q^2) - \frac{k}{4\pi}\Pi_o(q^2)\epsilon_{\mu\nu\lambda}q^\lambda\right\} \tag{3.14}$$

This result follows from the Ward-Takahashi identities of the gauge theory. As pointed out in [29], radiative corrections from the scalar loop induce a finite renormalization of the charge parameter $e^2$ which leads to a renormalized charge $e_r^2(e^2)$ which is finite as $e^2 \to \infty$. It is therefore more natural to consider not the anyon limit of the topologically massive gauge theory, but the equivalent limit $q \to 0$ of small momentum transfer. As has been discussed extensively in [25]–[27], the topologically massive gauge theory is infrared finite in the Landau gauge and so all renormalized quantities are non-singular at $q = 0$. In fact, at least to 2-loop order, all renormalization constants are independent of momenta and coincide exactly with their zero momentum limit. Substituting (3.14) into (3.13) and taking this limit, its imaginary part becomes[2]

$$\lim_{q\to 0}\ \mathrm{Im}\ \mathcal{G}(p_1, p_2; q) = -\frac{16\pi\dim(G)T_R(G)}{k}\ \Pi_o(0)\ \frac{\epsilon_{\mu\nu\lambda}p_1^\mu p_2^\nu q^\lambda}{q^2} \tag{3.15}$$

which leads to the one-loop vacuum polarization contribution

$$\Delta^{(1)}_{\rm vac} = -\frac{T_R(G)}{k}\ \Pi_o(0) \tag{3.16}$$

to the induced spin of the scalar fields.

Consider first the contribution from the scalar loop and the scalar tadpole. They are

[2]Radiative corrections also introduce a finite contribution in the anyon limit to the real part of the amplitude which can be attributed to the finite renormalization of the charge parameter $e^2$ and a short-ranged Pauli interaction between the magnetic fluxes carried by the anyons that are induced by the Chern-Simons gauge field [29, 30].

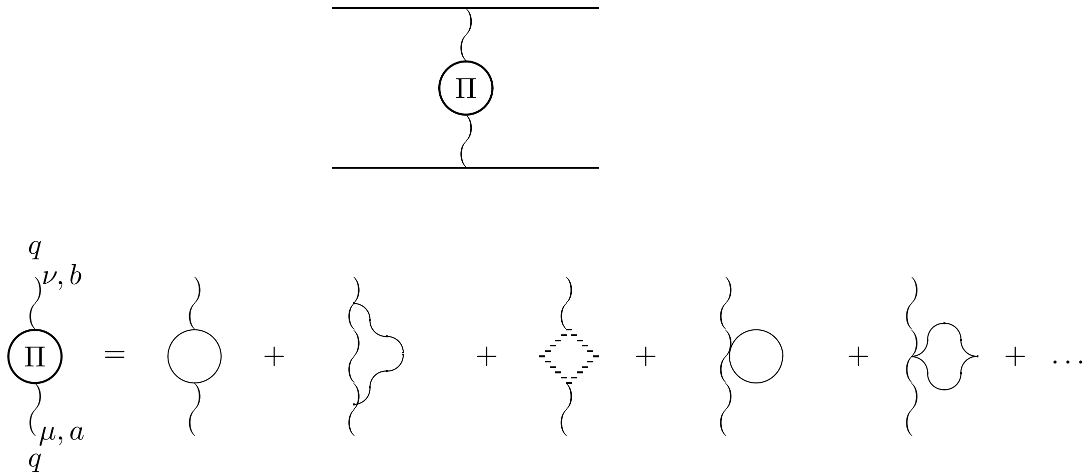



**Figure 3.2:** Total one-loop vacuum polarization contribution to the amplitude. The gluon self-energy is $\Pi^{ab}_{\mu\nu}(q)$, and the dashed lines denote the Faddeev-Popov ghost fields.

given, respectively, by

$$\Pi^{(s)ab}_{\mu\nu}(q) = -\int \frac{d^3k}{(2\pi)^3}\ \Big\{\mathrm{tr}\ (2k-q)_\mu R^a (2k-q)_\nu R^b S(k)S(k-q) + 2ig_{\mu\nu}\ \mathrm{tr}\ R^a R^b S(k)\Big\} \tag{3.17}$$

which can be evaluated in the infrared limit to give [31]

$$\lim_{q\to 0} \Pi^{(s)ab}_{\mu\nu}(q) = -i\delta^{ab}\frac{T_R(G)}{24\pi m} q^2 g^{\perp}_{\mu\nu}(q) \tag{3.18}$$

The correction (3.18) renormalizes the charge parameter to $1/e_r^2 = 1/e^2 + T_R(G)/24\pi m$ which has the finite value $e_r^2(e^2=\infty) = 24\pi m/T_R(G)$ in the anyon limit, as mentioned above[3]. The polarization tensor (3.17) has only a parity-even part, $\Pi^{(s)}_o(q^2) = 0$, which owes to the well-known fact that the scalar loop does not renormalize the Chern-Simons coefficient $k$ [25, 31]. It therefore does not contribute to the Aharonov-Bohm part of the amplitude (3.13). Next, consider the contributions from the gluon and ghost loops, and the gluon tadpole diagram, respectively,

$$\begin{aligned}\Pi^{(g)ab}_{\mu\nu}(q) \ &= \ \int \frac{d^3k}{(2\pi)^3}\ \Big\{\Gamma^{afe}_{\mu\lambda\alpha}(-q,k,q-k)\Gamma^{bcd}_{\nu\rho\beta}(q,-k,k-q)G^{\lambda\rho}_{fc}(k)G^{\beta\alpha}_{de}(k-q) \\ &\quad + \tilde{\Gamma}^{efa}_{\mu}(k-q,k;q)\tilde{\Gamma}^{cdb}_{\nu}(k,k-q;q)\tilde{G}^{fc}(k)\tilde{G}^{de}(k-q) + \Gamma^{abcd}_{\mu\nu\lambda\rho}(q,q,k,k)G^{\lambda\rho}_{cd}(k)\Big\}\end{aligned} \tag{3.19}$$

[3]Strictly speaking, we should take the limit $m\to\infty$ where the mesons become fixed external sources. As we shall see, the associated conformal weights are independent of any mass parameters of the model and thus in this limit we would obtain the same results. The anyon limit can then be reached in this regime even in the renormalized theory.

The only parity-odd contribution comes from the parity-odd part of the first term in (3.19) (from the gluon loop). To find it, we contract (3.19) with $\frac{k}{4\pi}\frac{\epsilon_{\mu\nu\lambda}q^\lambda}{2q^2}$, and after some algebra we get

$$\Pi_o^{(g)}(q^2) = -\frac{4\pi}{k}\frac{C_2(G)}{q^2}M\int\frac{d^3k}{(2\pi)^3}\,\frac{[k^2q^2-(k\cdot q)^2][5k^2+5(k\cdot q)+4q^2+2M^2]}{k^2(k^2-M^2)(k+q)^2[(k+q)^2-M^2]} \tag{3.20}$$

The integral in (3.20) is convergent and can be evaluated explicitly using standard methods [25, 27, 32]. Doing so, and then taking the low-energy limit we find

$$\Pi_o(0) = \frac{7}{3k}C_2(G)\ \mathrm{sgn}(k) \tag{3.21}$$

which gives

$$\Delta^{(1)}_{\rm vac} = -\frac{7}{3k^2}T_R(G)C_2(G)\ \mathrm{sgn}(k) \tag{3.22}$$

Next, we consider the 1-loop corrections to the meson-meson-gluon vertex (Fig. 3.3), which lead to the amplitude

$$\mathcal{V}(p_1,p_2;q) = \ \mathrm{tr}\ R^a(2p_1-q)^\mu R^b\Gamma^\nu(p_2,-q)G^{ab}_{\mu\nu}(q) + \ \mathrm{tr}\ R^a\Gamma^\mu(p_1,q)R^b(2p_2+q)^\nu G^{ab}_{\mu\nu}(q) \tag{3.23}$$

where $\Gamma_\mu(p,q)$ is the irreducible vertex function which by gauge invariance is given in terms of renormalization constants as

$$\Gamma_\mu(p,q) = -i(Z_e-1)(2p-q)_\mu - \frac{2i}{m}(Z_o-1)\epsilon_{\mu\nu\lambda}p^\nu q^\lambda \tag{3.24}$$

Here $Z_e$ is the meson charge renormalization constant. The only relevant contributions in the amplitude (3.23) will be from the longitudinal part of (3.24) contracted with the partity-odd part of the gauge propagator, or else from a parity-odd part of (3.24) which is singular at $q=0$ contracted with the parity-even part of the gauge propagator. By the infrared finiteness of the topologically massive gauge theory in the Landau gauge, this latter possibility does not occur. This can also be checked explicitly by examining the infrared structure of the Feynman integrals involved. For instance, it was shown in [29] that the first, triangle graph in Fig. 3.3 gives $Z_o^{(1)} = 1/k$ at one-loop order which is just the induced fractional spin of the anyons leading to an anomalous Pauli magnetic moment interaction. This piece is therefore non-singular at $q=0$ and does not contribute to the Aharonov-Bohm scattering. The former, longitudinal contribution is easily found by exploiting the Slavnov-Taylor identities for the non-abelian gauge theory which express the gauge invariance of the renormalized quantum field theory through the universality of the gauge coupling. The relation we shall use relates the meson-gauge 3-point coupling to the ghost-gauge coupling.

Consider the meson self energy operator $\Sigma(p)$ (Fig. 3.4) which renormalizes the bare meson propagator as $S(p)^{-1}\to S(p)^{-1}+\Sigma(p)$. The self-energy can be expressed in terms of the scalar wavefunction renormalization constant $Z_\phi$ as

$$\Sigma(p) = -i(Z_\phi-1)(p^2-m^2) \tag{3.25}$$

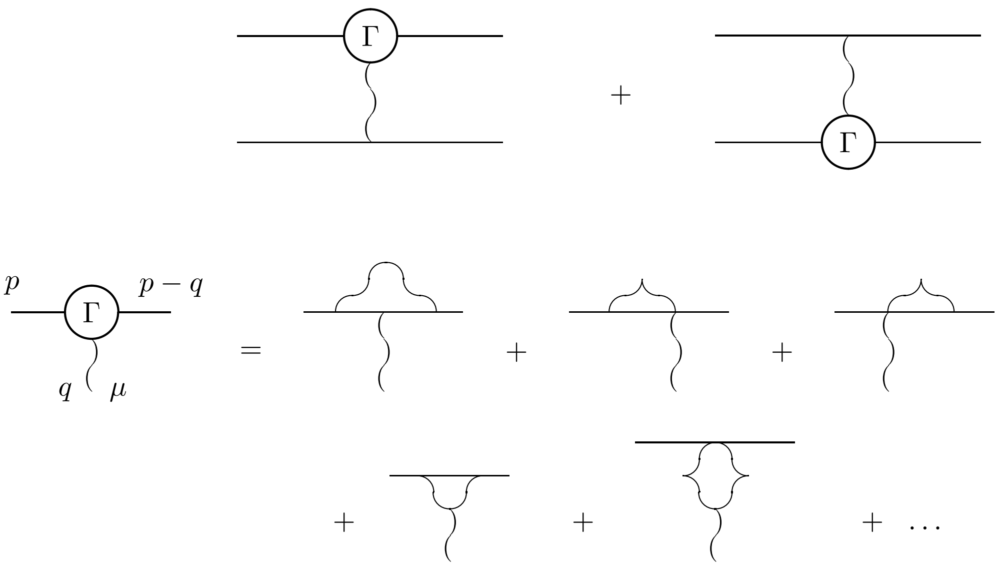



**Figure 3.3:** Total one-loop vertex correction to the scattering amplitude. The proper vertex function is $\Gamma_\mu(p,q)$.

Similarly, we introduce the ghost self-energy operator $\tilde{\Pi}^{ab}(p) = \delta^{ab}\tilde{\Pi}(p)$ which renormalizes the bare ghost field propagator as $\tilde{G}(p)^{-1} \to \tilde{G}(p)^{-1} + \tilde{\Pi}(p)$ and which can be written in terms of the ghost wavefunction renormalization constant $\tilde{Z}$ as

$$\tilde{\Pi}(p) = -ip^2(\tilde{Z} - 1) \tag{3.26}$$

Finally, we introduce the renormalized ghost-ghost-gluon vertex by

$$\tilde{\Gamma}^{(r)abc}_{\mu}(p) = -if^{abc}(\tilde{Z}_g - 1)p_\mu \tag{3.27}$$

where $\tilde{Z}_g$ is the ghost charge renormalization constant. The desired Slavnov-Taylor identity is then

$$\frac{Z_\phi}{Z_e} = \frac{\tilde{Z}}{\tilde{Z}_g} \tag{3.28}$$

and it expresses the fact that the mesons and ghosts couple in the same way to the non-abelian gauge field $A$. As shown in [25, 27], $\tilde{Z}_g = 1$ at least to two-loop order in perturbation theory. At one-loop order, we write the renormalization constants as $Z = 1 + Z^{(1)}$, where $Z^{(n)}$ denotes the order $1/k^n$ perturbative contribution. Then the Slavnov-Taylor identity (3.28) at one-loop order is

$$Z_e^{(1)} = Z_\phi^{(1)} - \tilde{Z}^{(1)} \tag{3.29}$$

or, by the definitions of the renormalization constants, we have the identity

$$q^\mu\Gamma_\mu^{(1)}(p,q) + \Sigma^{(1)}(p-q) - \Sigma^{(1)}(p) = \tilde{\Pi}^{(1)}(p) - \tilde{\Pi}^{(1)}(p-q) \tag{3.30}$$

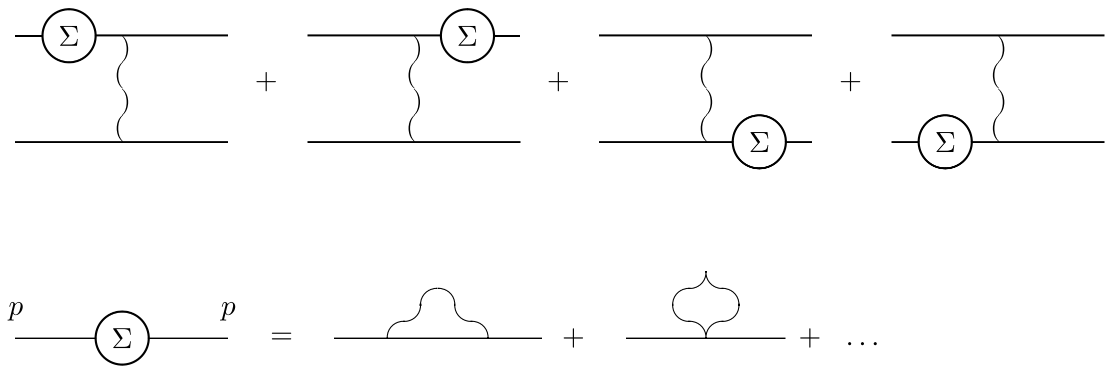


**Figure 3.4:** Total one-loop meson self-energy contribution to the scattering amplitude. The self-energy operator is $\Sigma(p)$.

at one-loop order.

The identity (3.30) shows that the total sum of the one-loop vertex corrections shown in Fig. 3.3 and the meson self-energy corrections in Fig. 3.4 can be reduced simply to the calculation of the ghost self-energy (Fig. 3.5). In fact, it shows that the longitudinal parts of the contributions from the 3-gluon vertex exchange amplitudes (fourth graph in Fig. 3.3) are determined entirely by the ghost wavefunction renormalization constant, while the other longitudinal vertex and self-energy corrections cancel each other out. This is because, first of all, the meson-meson-gluon-gluon vertex is independent of momentum and so the one-loop vertex function containing this vertex and the 3-gluon vertex (fifth diagram in Fig. 3.3) is a function only of $q$, and hence is proportional to $q^\mu$. In the Landau gauge, the amplitudes involving this vertex function therefore vanish by transversality of the gluon propagator, $q^\mu G^{ab}_{\mu\nu}(q) = 0$. Secondly, modulo a common group theoretical factor, the first three vertex functions in Fig. 3.3 and the scalar self-energy operators are the same as those in the abelian theory ($G = U(1)$) where the Slavnov-Taylor identity (3.28) is replaced by the simpler Ward identity $Z_\phi = Z_e$ (as then $f^{abc} = 0$ in (3.1) and so the ghost field decouples from the gauge field). This means that these vertex and self-energy corrections cancel each other. At one-loop order therefore, the 3-gluon vertex renormalization constant $Z^{(1)}_{e,3}$ is determined entirely from the ghost wavefunction renormalization constant as

$$Z^{(1)}_{e,3} = -\tilde{Z}^{(1)} \tag{3.31}$$

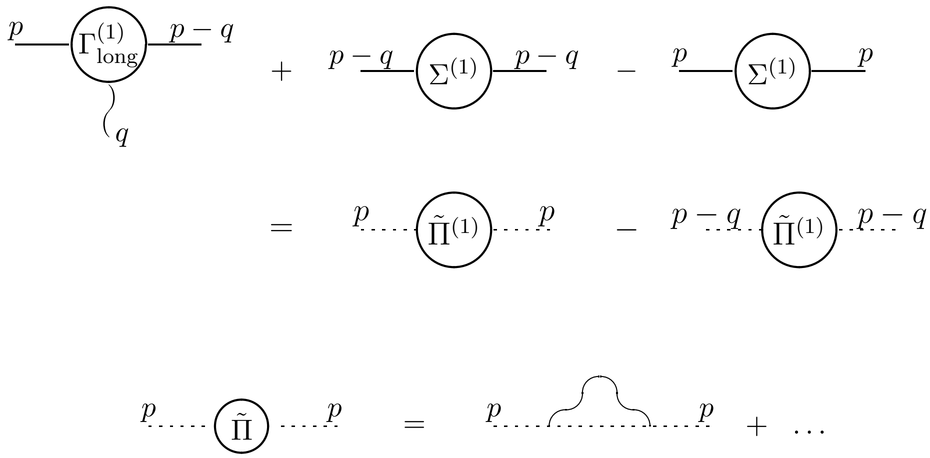


**Figure 3.5:** The Slavnov-Taylor identity expressing the universality of the gauge field coupling to the meson and ghost fields at one-loop order. $\tilde{\Pi}(p)$ is the ghost self-energy operator.

The ghost self-energy is given by

$$
\begin{aligned}
\tilde{\Pi}^{ab}(p) &= \int \frac{d^3k}{(2\pi)^3}\ \tilde{\Gamma}^{ace}_{\mu}(-p, p-k; k)\tilde{\Gamma}^{dbf}_{\nu}(k-p, p; -k)\tilde{G}^{cd}(p-k)G^{\mu\nu}_{ef}(k) \\
&= \frac{4\pi i}{k}\delta^{ab}p^2 C_2(G) M \int \frac{d^3k}{(2\pi)^3}\ \frac{(k\cdot p)^2 - k^2p^2}{k^2(k+p)^2(k^2-M^2)}
\end{aligned}
\tag{3.32}
$$

Again the Feynman integrations can be carried out using standard methods [25, 27, 32], and in the anyon limit we get

$$
\lim_{p\to 0}\tilde{\Pi}^{ab}(p) = \frac{2i}{3k}p^2\delta^{ab}C_2(G)\ \mathrm{sgn}(k) \tag{3.33}
$$

which leads to

$$
Z^{(1)}_{e,3} = \frac{2}{3k}C_2(G)\ \mathrm{sgn}(k) \tag{3.34}
$$

The contribution to the imaginary part of the amplitude (3.23) from the self-energy and longitudinal vertex corrections in the anyon limit is therefore given by

$$
\lim_{q\to 0}\ \mathrm{Im}\ \mathcal{V}_{\rm long}(p_1, p_2; q) = \frac{32\pi}{k}T_R(G)\dim(G)Z^{(1)}_e\frac{\epsilon_{\mu\nu\lambda}p_1^{\mu}p_2^{\nu}q^{\lambda}}{q^2} \tag{3.35}
$$

which leads to the conformal weight

$$
\Delta^{(1)}_{\rm long} = \frac{4}{3k^2}T_R(G)C_2(G)\ \mathrm{sgn}(k) \tag{3.36}
$$

Finally, consider the ladder diagrams depicted in Fig. 3.6. The corresponding Feynman

integrations for the first, fourth and fifth diagrams there are

$$
\begin{aligned}
A_{\Box} &= i \int \frac{d^3k}{(2\pi)^3} \; \mathrm{tr}\; (2p_1 - q - k)^\mu R^a (2p_2 + q - k)^\nu R^b (2p_1 - k)^\lambda R^c (2p_2 + k)^\rho R^d \\
&\qquad\qquad \times S(p_1 - k) S(p_2 + k) G^{ab}_{\nu\mu}(k - q) G^{cd}_{\lambda\rho}(k) \\
A_{\triangle} &= \int \frac{d^3k}{(2\pi)^3} \; \mathrm{tr}\; (2p_2 + q + k)^\nu R^a (2p_2 + k)^\rho R^b g^{\mu\lambda} \{R^c, R^d\} S(p_2 + k) G^{ac}_{\mu\nu}(q - k) G^{bd}_{\lambda\rho}(k) \\
A_{\bigcirc} &= -i \int \frac{d^3k}{(2\pi)^3} \; \mathrm{tr}\; g^{\mu\lambda} g^{\nu\rho} \{R^a, R^b\}\{R^c, R^d\} G^{ac}_{\mu\nu}(q - k) G^{bd}_{\lambda\rho}(k)
\end{aligned}
\tag{3.37}
$$

The second "box" diagram in Fig. 3.6 is then given by $A_{\Box}(p_2 \to -(p_2 + q))$ while the third "triangle" graph is $A_{\triangle}(p_2 \to p_1, q \to -q)$. The structure of the Feynman integrals in (3.37) has been examined extensively in [30] where it was shown that each of these ladder amplitudes vanishes in the anyon limit.

The total conformal weight at one-loop order is therefore completely determined by the one-loop renormalization of the Chern-Simons coefficient $k$ and is given as the sum of the weights from the pure gauge vacuum polarization loop and the longitudinal part of the 3-gluon vertex corrections. Thus the induced spin up to one-loop order is

$$
\Delta^{(1)} = \Delta^{(0)} + \Delta^{(1)}_{\rm vac} + \Delta^{(1)}_{\rm long} = \frac{T_R(G)}{k} - \frac{1}{k^2} T_R(G) C_2(G)\ \mathrm{sgn}(k) \tag{3.38}
$$

which coincides with the leading orders in the asymptotic expansion (2.10) of the Knizhnik-Zamolodchikov formula (2.7). The above calculation can also be for the most part extended to 2-loop order using the 2-loop pure gauge self-energies calculated in [32]. However, although it is conjectured that the ladder amplitudes vanish to all loop-orders in the anyon limit [30], an explicit demonstration of such a non-renormalization has yet to be established.

Note that the anomalous spin here depends on the sign of the Chern-Simons coefficient $k$. Transformations which change the orientation of the 3-manifold $\mathcal{M}$ on which the Chern-Simons gauge theory is defined (such as parity or time-reversal) change the sign of the Chern-Simons term (1.2). The dependence of the perturbative weights on $\mathrm{sgn}(k)$ ensures that one can compensate in the effective action a sign reversal of the Chern-Simons term by a change in sign of $k$ thus rendering the quantum field theory (1.1) invariant under orientation-reversing isometries of $\mathcal{M}$. This feature is necessary to maintain the covariance of both the topologically massive and topological Chern-Simons quantum gauge theories. It is also consistent with the orientation-reversing features of the induced WZNW model on $\partial\mathcal{M}$ which change the sign of the central charge of the corresponding Kac-Moody algebra. The relative sign differences between the coefficients of several Chern-Simons theories is important for the corresponding induced current algebras where a negative level coefficient would lead to a non-unitary model.

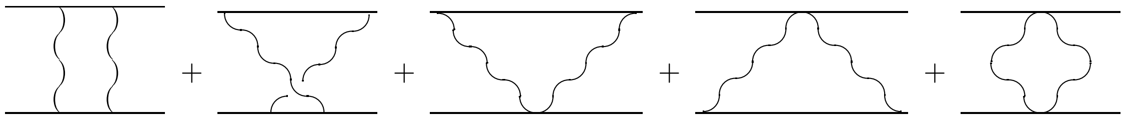


**Figure 3.6:** One-loop ladder (2-gluon exchange) diagram contribution.

## 3.2 Coupling to Spinor Fields and the Supersymmetric WZNW Model

It is possible to carry out the same calculation as above when instead of mesons we couple fermions to topologically massive Yang-Mills theory with the action

$$S_f = \int d^3x \ \left\{\bar{\psi}_A i\gamma^\mu \left(\partial_\mu\psi_A - iR^a_{AB}A^a_\mu\psi_B\right) - m_f\bar{\psi}_A\psi_A\right\} \tag{3.39}$$

where $\psi_A$ are 2-component Dirac fields and the (2 + 1)-dimensional gamma-matrices can be represented by Pauli spin matrices, $\gamma^0 = \sigma^3, \gamma^1 = i\sigma^1, \gamma^2 = i\sigma^2$ [30]. In a 3-dimensional spacetime, the fermion mass $m_f$ may be positive or negative for parity-odd fermion fields. Now the fermion-fermion-gluon vertex is $-i\gamma_\mu R^a_{AB}$, while the free fermion propagator in momentum space is

$$S^{(f)}_{AB}(p) = \langle\bar{\psi}_A(p)\psi_B(-p)\rangle_\psi = i(p_\mu\gamma^\mu - m_f)^{-1}\delta_{AB} \tag{3.40}$$

All of the analysis carries through as above for scalars using standard identities for Dirac spinors and fermion bilinears in (2 + 1)-dimensions [30], the only differences being that there are no Feynman diagrams involving a fermion-fermion-gluon-gluon vertex, and that parity-odd fermions renormalize the Chern-Simons coefficient $k$ at one-loop order [25, 27]. The one-fermion loop contribution to the vacuum polarization tensor is

$$\Pi^{(f)ab}_{\mu\nu}(q) = \int \frac{d^3k}{(2\pi)^3} \ \mathrm{Tr}\ R^a\gamma_\mu S^{(f)}(k)R^b\gamma_\nu S^{(f)}(k-q) \tag{3.41}$$

where the trace is over both spinor and group indices. It was shown in [33] that this contribution in the infrared limit is

$$\lim_{q\to 0}\Pi^{(f)ab}_{\mu\nu}(q) = -\delta^{ab}\left\{\frac{iT_R(G)}{12\pi|m_f|}q^2 g^\perp_{\mu\nu}(q) + \frac{\mathrm{sgn}(m_f)T_R(G)}{4\pi}\epsilon_{\mu\nu\lambda}q^\lambda\right\} \tag{3.42}$$

The extra parity-odd structure arises from contraction of the spinor vertices because in (2 + 1)-dimensions the gamma-matrices obey $\gamma_\mu\gamma_\nu = 2g_{\mu\nu} + i\epsilon_{\mu\nu\lambda}\gamma^\lambda$ [30]. This gives the extra contribution $\Pi^{(f)}_o(0) = T_R(G)\ \mathrm{sgn}(m_f)/k$ to (3.21), and thus the Knizhnik-Zamolodchikov

formula up to 1-loop order becomes

$$\Delta_f^{(1)} = \frac{T_R(G)}{k} - \frac{T_R(G)}{k^2}\left(C_2(G)\ \mathrm{sgn}(k) + T_R(G)\ \mathrm{sgn}(m_f)\right) \tag{3.43}$$

The formula (3.43) agrees with the leading orders of the asymptotic expansion of the conformal dimension

$$\Delta_f = \frac{T_R(G)}{k + C_2(G)\ \mathrm{sgn}(k) - T_R(G)\ \mathrm{sgn}(m_f)} \tag{3.44}$$

Now an interesting feature immediately arises. If we couple the topologically massive gauge theory to Dirac fermions which transform under the adjoint representation of $G$, so that $T_R(G) = -C_2(G)$, then we can adjust the mass parameter of the fermions so that the one-loop contribution to the induced spin vanishes and (3.44) coincides exactly with the tree-level result. In fact, we can adjust the fermion mass $m_f$ and rescale the Grassmann fields so that the spinor-coupled gauge theory coincides with the supersymmetric topologically massive gauge theory which is defined by the bulk action [8, 34, 35]

$$\begin{aligned} \mathcal{S} \;=\; & S_{TMGT} + \frac{1}{2e^2}\int d^3x\ \bar{\chi}^a i\gamma^\mu\left(\partial_\mu\chi^a + if^{abc}A_\mu^b\chi^c\right) - \frac{k}{8\pi}\int d^3x\ \bar{\chi}^a\chi^a \\ & + \frac{k}{8\pi}\int d^3x\ i\bar{\lambda}^a\left(\gamma^\mu\partial_\mu\chi^a - \frac{1}{3}f^{abc}\gamma^\mu\partial_\mu\lambda^b L^c - \frac{2}{3}f^{abc}\epsilon^{\mu\nu\rho}\gamma_\nu\partial_\mu\lambda^b A_\rho^c\right) \end{aligned} \tag{3.45}$$

where $\chi^a$ are Majorana fermion fields in the adjoint representation of $G$, and $L^a$ are auxilliary scalar fields with $\lambda^a$ their (Majorana) spinor superpartner fields. The Majorana representation of the $(2+1)$-dimensional Dirac algebra can be taken to be $\gamma^0 = \sigma^2$, $\gamma^1 = i\sigma^1$, $\gamma^2 = i\sigma^3$, and the Majorana spinors $\chi^a = i\sigma^2(\bar{\chi}^a)^T$ have 2 real components.

The action (3.45) is invariant (up to surface terms) under the infinitesimal $N=1$ supersymmetry transformation

$$\begin{aligned} &\delta A_\mu = -\bar{\varepsilon}\gamma_\mu\chi + \bar{\varepsilon}\partial_\mu\lambda \quad , \quad \delta\chi = \epsilon^{\mu\nu\rho}\partial_\nu A_\rho\gamma_\mu\varepsilon - \gamma^\nu\partial_\nu(L + \gamma^\mu A_\mu)\varepsilon \\ &\delta L = -\bar{\varepsilon}\chi + \bar{\varepsilon}\gamma^\mu\partial_\mu\lambda \quad , \quad \delta\lambda = A\varepsilon + \gamma^\mu A_\mu\varepsilon \end{aligned} \tag{3.46}$$

where $\varepsilon$ is a global, complex-valued Grassmann parameter. The calculation of the one-fermion loop contribution for the Majorana spinor fields $\chi^a$ is then identical to that above, because in $(2+1)$-dimensions the Majorana propagator is identical to the Dirac one and thus the Dirac and Majorana fermion loop contributions are the same. The exact (tree-level) conformal dimension $\Delta_{SUSY} = C_2(G)/k$ coincides with that of the $N=1$ supersymmetric WZNW model [24] which is induced by (3.45) on the boundary $\partial\mathcal{M}$ [35]. Its exactness at tree-level is a consequence of the standard non-renormalizations in supersymmetric field theories where the supersymmetry leads to mutual cancellations between the bosonic and fermionic loops in perturbation theory. Notice that for the abelian theory where $G = U(1)$, we have $C_2(G) = 0$ and the exactness of the scalar conformal weight $\Delta_{U(1)} = 1/k$ at tree-level follows from the

well-known non-renormalization properties of abelian Chern-Simons gauge theory [31, 33]. For the supersymmetric abelian theory, the vanishing of this weight is a consequence of the fact that in the abelian theory $f^{abc} = 0$ in (3.45) and so all fields are decoupled from the gauge field in this case.

# 4 Effects of Gravitational Dressing: Conformal Weights in Topologically Massive Gravity

In this Section we shall assume that the spacetime $\mathcal{M}$ has a dynamical metric $g$ with action (1.6) and examine the effects of gravitational dressing on the anomalous spin values of the last Section. As with the derivation of Liouville theory from topologically massive gravity [18, 19], we shall find it convenient to study the model (1.6) in the first order formalism for general relativity[4] [36]. We therefore introduce the dreibein fields $e^a = e^a_\mu dx^\mu$ which span the frame bundle of $\mathcal{M}$. Here and in the following greek letters will label the spacetime indices (i.e. the components of the local basis vectors of the tangent space) and latin letters will denote the basis index of the local Lorentz group $SO(2,1)$ of the tangent bundle. The dreibein fields are related to the metric $g$ of $\mathcal{M}$ by the orthonormality condition $g^{\mu\nu}e^a_\mu e^b_\nu = \eta^{ab} = \mathrm{diag}(1,-1,-1)$, or equivalently by the completeness relation

$$\eta_{ab}e^a_\mu e^b_\nu = g_{\mu\nu} \tag{4.1}$$

The action (1.6) for topologically massive gravity in the first order formalism is

$$S_{TMG} = S_E + S^{(G)}_{CS} + S_\lambda \tag{4.2}$$

where

$$S^{(G)}_{CS} = \frac{k'}{8\pi}\int_{\mathcal{M}}\left(\omega^a \wedge d\omega^a + \frac{2}{3}\epsilon^{abc}\omega^a \wedge \omega^b \wedge \omega^c\right) \tag{4.3}$$

is the gravitational Chern-Simons action, and $\omega^a = \epsilon^{abc}\omega^{bc}$ with $\omega^{ab} = -\omega^{ba} = \omega^{ab}_\mu dx^\mu$ the spin-connection of the frame bundle of $\mathcal{M}$. The Einstein action is

$$S_E = \kappa\int_{\mathcal{M}} e^a \wedge R^a \tag{4.4}$$

where

$$R^a = R^a_{\mu\nu}dx^\mu \wedge dx^\nu = d\omega^a + \epsilon^{abc}\omega^b \wedge \omega^c \tag{4.5}$$

[4]It is also possible to carry out this analysis when the gravitational Chern-Simons action is expressed in terms of Christoffel connections rather than spin connections [20].

is the curvature of the spin-connection $\omega^a$. We have also included in (4.2) the Lagrange multiplier term

$$S_\lambda = \int_{\mathcal{M}} \lambda^a \wedge \left( de^a + 2\epsilon^{abc}\omega^b \wedge e^c \right) \tag{4.6}$$

where the Lagrange multiplier fields $\lambda^a = \lambda^a_\mu dx^\mu$ enforce the constraint which ensures that the usual covariant derivative $\nabla$ constructed from the spin-connection is compatible with the metric $g$ (i.e. $\nabla e^a = 0$) so that the Einstein action in the first order formalism coincides with the usual one of general relativity.

The ordinary, pure Einstein theory in three-dimensions contains no propagating degrees of freedom and is a topological field theory. The addition of the gravitational Chern-Simons term makes the gravitons of the theory massive with topological mass (1.7). In the infrared limit $\mu \to \infty$ (equivalently $\kappa \to \infty$) this propagating degree of freedom decouples and the Chern-Simons term in (4.2) becomes irrelevant (note that this is opposite to the situation in topologically massive Yang-Mills theory). Then the kinetic term for the Liouville field in (1.8) vanishes and the gravity theory (4.2) induces the theory on $\partial\mathcal{M}$ which describes the moduli space of Riemann surfaces. The calculation of the conformal weights as in the last Section in this infrared limit will be associated with those of string theory in the critical dimension when the induced anomaly vanishes. As before, we shall regulate the gravitational field theory in the following by computing everything with the coupling to the topologically massive graviton field $e^a_\mu$, and then take this infrared limit to recover the observables of the critical string theory. The effective coupling constant of the topologically massive gravity theory is the super-renormalizable, dimensionless expansion parameter $\mu/\kappa \sim 1/k'$. The topological graviton mass regulates the infrared divergences of the Einstein theory and, based on the naive power counting renormalizability of topologically massive gravity [37, 38], we expect all renormalized quantities to be non-singular.

The action (4.2) is diffeomorphism invariant (i.e. generally covariant) and it also possesses a local $SO(2,1)$-invariance defined by rotations of the dreibein fields. In this context, the model can be viewed as an $SO(2,1)$ gauge theory for the spin-connection $\omega^a$. Both the dreibein and Lagrange multiplier fields above transform under the adjoint representation of this gauge group. With this point of view, we will study the model (4.2) perturbatively by expanding the graviton field about a flat background metric, i.e. we shift the dreibein fields as

$$e^a_\mu \to e^a_\mu + \delta^a_\mu \quad , \tag{4.7}$$

and view the topologically massive gravity theory as a quantum field theory on a flat space. Introducing the new variables $\beta^a$ defined by

$$\beta^a = \lambda^a + \kappa\omega^a \tag{4.8}$$

the topologically massive gravity action becomes

$$S_{TMG} = \int d^3x \, \Big\{ \epsilon^{\mu\nu\lambda}\beta^a_\mu \partial_\nu e^a_\lambda + 2(\beta^\mu_\mu \omega^\nu_\nu - \beta^\mu_\nu \omega^\nu_\mu) + \frac{k'}{8\pi}\epsilon^{\mu\nu\lambda}\omega^a_\mu \partial_\nu \omega^a_\lambda - \kappa(\omega^\mu_\mu \omega^\nu_\nu - \omega^\mu_\nu \omega^\nu_\mu) \\ + \epsilon^{\mu\nu\lambda}\left(2\epsilon^{abc}\beta^a_\mu \omega^b_\nu e^c_\lambda - \kappa\epsilon^{abc}e^a_\mu \omega^b_\nu \omega^c_\lambda + \frac{k'}{12\pi}\epsilon^{abc}\omega^a_\mu \omega^b_\nu \omega^c_\lambda\right)\Big\} \tag{4.9}$$

where here and in the following all repeated indices are understood to be summed over by contracting with the flat Minkowski metric $\eta^{ab}$. To obtain the Feynman rules, we note first of all that it is not necessary to fix the gauge for the spin-connection $\omega^a$, because its quadratic form in (4.9) is non-degenerate (due to the topological mass term). Thus we fix the gauge only for the $e$ and $\beta$ fields, which is done by introducing a Faddeev-Popov ghost action (3.1) for each of them with $f^{abc} = \epsilon^{abc}$ and $A = e$ or $\beta$ there. The quadratic part of the action (4.9) is then

$$S_2 = \int d^3x \, \Big\{ \epsilon^{\mu\nu\lambda}\beta^a_\mu \partial_\nu e^a_\lambda + \frac{1}{2\xi_e}(\partial^\mu e^a_\mu)^2 + \frac{1}{2\xi_\beta}(\partial^\mu \beta^a_\mu)^2 + \frac{k'}{8\pi}\epsilon^{\mu\nu\lambda}\omega^a_\mu \partial_\nu \omega^a_\lambda \\ - \kappa(\omega^\mu_\mu \omega^\nu_\nu - \omega^\mu_\nu \omega^\nu_\mu) + 2(\beta^\mu_\mu \omega^\nu_\nu - \beta^\mu_\nu \omega^\nu_\mu)\Big\} \tag{4.10}$$

where $\xi_e$ and $\xi_\beta$ are gauge fixing parameters. As before, the simplest gauge choice is the transverse Landau gauge wherein $\xi_e = \xi_\beta = 0$ and the propagators of the fields are all transverse. In what follows we shall choose this gauge.

To transform quantities to momentum space, we make the replacements $\partial_\mu \to ip_\mu$ in (4.10). Then the $e\beta$ propagator is

$$-\, i\langle \beta^i_\mu(p) e^j_\nu(-p)\rangle_{e\beta} = -i\eta^{ij}\epsilon_{\mu\nu\lambda}p^\lambda/p^2 \tag{4.11}$$

The propagator for the spin connection

$$-\, i\left\langle \omega^i_\mu(p)\omega^j_\nu(-p)\right\rangle_\omega = \frac{4\pi}{k'}\ \Omega^{ij}_{\mu\nu}(p) \tag{4.12}$$

is determined as the solution of the equation

$$(-i\epsilon_{\mu\rho\lambda}p^\lambda \eta^{ik} - \mu X^{ik}_{\mu\rho})\ \Omega^{kj}_{\rho\nu}(p) = \eta^{ij}\eta_{\mu\nu} \tag{4.13}$$

where

$$X^{kl}_{\lambda\rho} = \delta^k_\lambda \delta^l_\rho - \delta^l_\lambda \delta^k_\rho \tag{4.14}$$

Inverting the equation (4.13), we find after some algebra

$$
\begin{aligned}
\Omega^{ij}_{\mu\nu}(p) &= \frac{\mu(\Lambda^i_\mu(p)\Lambda^j_\nu(p) - \Lambda_{\mu\nu}(p)\Lambda^{ij}(p) - \Lambda^j_\mu(p)\Lambda^i_\nu(p))}{2(p^2-\mu^2)} - \frac{(\Lambda_{\mu\nu}(p)\Lambda^{ij}(p) - \Lambda^j_\mu(p)\Lambda^i_\nu(p))}{2\mu} \\
&- \frac{1}{2\mu^3}(\Lambda_{\mu\nu}(p)p^ip^j + \Lambda^{ij}(p)p_\mu p_\nu - \Lambda^j_\mu(p)p^ip_\nu - \Lambda^i_\nu(p)p_\mu p^j) \\
&- \frac{i(\epsilon_{\mu\nu\lambda}p^\lambda\Lambda^{ij}(p) + \epsilon^{ijk}p_k\Lambda_{\mu\nu}(p))}{2(p^2-\mu^2)} \\
&- \frac{i}{2\mu^2}\left(\epsilon^{i\lambda}{}_\mu p_\lambda \frac{p_\nu p^i}{p^2} + \epsilon_\nu{}^{\lambda j}p_\lambda \frac{p_\mu p^i}{p^2} + \epsilon_{\mu\lambda\nu}p^\lambda \frac{p^ip^j}{p^2} - \epsilon^{i\lambda j}p_\lambda \frac{p_\mu p_\nu}{p^2}\right)
\end{aligned}
\tag{4.15}
$$

where

$$
\Lambda_{\mu\nu}(p) = \eta_{\mu\nu} - p_\mu p_\nu/\mu^2 \tag{4.16}
$$

The $e\omega$ and pure graviton propagators can now be obtained from the momentum space convolutions

$$
-\,i\langle\omega^i_\mu(p)e^j_\nu(-p)\rangle_{e\omega} = 2\langle\omega^i_\mu(p)\omega^k_\lambda(-p)\rangle_\omega\ X^{kl}_{\lambda\rho}\ \langle\beta_\rho(p)e^j_\nu(-p)\rangle_{e\beta} \tag{4.17}
$$

$$
D^{ij}_{\mu\nu}(p) \equiv \langle e^i_\mu(p)e^j_\nu(-p)\rangle_e = -4\langle e^i_\mu(p)\beta^k_\lambda(-p)\rangle_{e\beta}\ X^{kl}_{\lambda\rho}\ \langle\omega^l_\rho(p)\omega^m_\alpha(-p)\rangle_\omega\ X^{mn}_{\alpha\sigma}\ \langle\beta^n_\sigma(p)e^j_\nu(-p)\rangle_{e\beta} \tag{4.18}
$$

which, using (4.11)–(4.16), after some algebra lead to

$$
\left\langle\omega^i_\mu(p)e^j_\nu(-p)\right\rangle_{e\omega} = -\frac{8\pi i}{k'\mu}\left(\frac{i\epsilon_{\mu\nu\lambda}p^\lambda}{p^2}\eta^{ij} - \Omega^{ij}_{\mu\lambda}(p)\delta^{\perp\lambda}_\nu(p)\right) \tag{4.19}
$$

$$
\begin{aligned}
D^{ij}_{\mu\nu}(p) &= \frac{16\pi i}{k'\mu^2}\left(\frac{\epsilon_{\mu\nu\lambda}p^\lambda}{p^2}\eta^{ij} + \delta^{\perp\rho}_\mu(p)\ \Omega^{ij}_{\rho\lambda}(p)\delta^{\perp\lambda}_\nu(p) + \frac{\mu}{p^2}\left(\delta^{\perp j}_\mu(p)\delta^{\perp i}_\nu(p) - \delta^{\perp i}_\mu(p)\delta^{\perp j}_\nu(p)\right)\right) \\
&= \frac{i}{\kappa}\left(\frac{\mu^2}{2p^2(p^2-\mu^2)}\left\{\left(\frac{p^2}{\mu^2}-2\right)\eta^\perp_{\mu\nu}(p)\eta^{\perp ij}(p) + \delta^{\perp i}_\mu(p)\delta^{\perp j}_\nu(p) + \delta^{\perp j}_\mu(p)\delta^{\perp i}_\nu(p)\right\}\right. \\
&\qquad \left. + \frac{i\mu}{4}\frac{p^\lambda}{p^2(p^2-\mu^2)}\left\{\epsilon_\mu{}^i{}_\lambda\delta^{\perp j}_\nu(p) + \epsilon_\mu{}^j{}_\lambda\delta^{\perp i}_\nu(p) + \epsilon_\nu{}^i{}_\lambda\delta^{\perp j}_\mu(p) + \epsilon_\nu{}^j{}_\lambda\delta^{\perp i}_\mu(p)\right\}\right)
\end{aligned}
\tag{4.20}
$$

The propagator (4.20) is the usual Deser-Yang graviton propagator [37] obtained from the second order formalism for topologically massive gravity. The parity-odd structure in (4.20) can lead to a gravitational analogue of the Aharonov-Bohm effect [39]. The Feynman rules for the interaction vertices $\omega^3$, $\omega^2 e$ and $\omega\beta e$ can be read off from the action (4.9).

In (2 + 1)-dimensions parity-odd matter fields (such as massive fermions or topologically massive vector bosons) induce gravitational Chern-Simons terms [40]. This term describes the central charge of the corresponding two-dimensional conformal field theory [4, 12, 15, 16, 18]. It was conjectured in [16] that the gravitational renormalization of $k'$ must coincide with

that predicted by the KPZ theory [17]. Here we shall demonstrate the equivalence between the KPZ (1 + 1)-dimensional results and the perturbative $1/k'$ expansion in topologically massive gravity at one-loop order by calculating the one-loop gravitational renormalization of the conformal dimension of some primary operators in the infrared limit $\kappa \to \infty$ when the topologically massive gravity theory becomes the topological Einstein one. As discussed in [19], it is only in the phase with non-zero vacuum expectation value for the dreibein field that one can treat the quantum field theory (4.9) perturbatively. In the topological phase where $\langle e_\mu^a \rangle = 0$, there is no background space-time and the quadratic approximation required to find the propagators does not exist. This is because there are only 2 gauge groups, i.e. diffeomorphisms and local $SO(2,1)$ or $SL(2,\mathbb{R})$ rotations, while all 3 of the fields $\beta$, $\omega$ and $e$ must be gauge-fixed. Thus we expect to get agreement with the KPZ theory only in the phase with $\langle e_\mu^a \rangle = \delta_\mu^a$. The topological phase may be related to the breakdown of the KPZ approach for $1 < c < 25$ strings.

We consider the minimal coupling (i.e. we assume that the spin connection is torsion free) of the topologically massive gravity theory above to matter. We are interested here in the gravitationally dressed scalar-coupled topologically massive gauge theory of the last Section. For simplicity, we consider only the abelian case. The results below generalize straightforwardly to the non-abelian case as well. Thus we are interested in coupling to the matter action

$$\begin{aligned} S_m[A,\phi,e] &= \int_{\mathcal{M}} d^3x \, \Big( -\frac{1}{4e^2}\sqrt{g}\; g^{\mu\lambda} g^{\nu\rho} F_{\mu\nu} F_{\lambda\rho} + \frac{k}{8\pi} \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \\ &\quad + \sqrt{g}\; g^{\mu\nu} [(\partial_\mu - iA_\mu)\phi]^* [(\partial_\nu - iA_\nu)\phi] - m^2 \sqrt{g}\; \phi^* \phi \Big) \end{aligned} \tag{4.21}$$

where $g = \det[g_{\mu\nu}]$ and $g^{\mu\nu} = \eta^{ab} e_a^\mu e_b^\nu$ with $e_a^\mu$ the inverse dreibein fields, i.e. $e_a^\mu e_\nu^a = \delta_\nu^\mu$. The ghost field terms play no role in the following and so will be ignored. The topologically massive gauge interaction is introduced to induce a non-zero bare spin $\Delta_0$ for the meson fields and also to study the interplay between gauge and gravity degrees of freedom. Otherwise, from the KPZ formula, we expect to obtain a dressed weight $\Delta = 0$. We shall see that the perturbative calculations in this matter-coupled topologically massive gravity theory are consistent with this. According to the Knizhnik-Zamolodchikov formula, the bare conformal spin of primary operators in the associated conformal field theory is

$$\Delta_0 = 1/k \tag{4.22}$$

We are interested in the one-loop order gravitational corrections to the tree-level Aharonov-Bohm amplitude depicted in Fig. 3.1, i.e. we wish to determine, to order $1/k'$, the transformation of the bare weight (4.22) due to the gravitational dressing from the associated Aharonov-Bohm amplitude as we did before. Upon shifting the dreibein fields as in (4.7),

besides the usual electrodynamical interactions there are infinitely many orders of the interactions between the graviton and the photon or meson in (4.21). This is because the $\sqrt{g}$ term when expanded in terms of the shifts (4.7) involves infinitely many powers of the dreibein field $e^a_\mu$. However, to compute the one-loop corrections to the diagram in Fig. 3.1, we need only consider those gravitational interactions involving at most 2 graviton fields $e^a_\mu$. Thus it suffices to expand the metric determinant factors in (4.21) to quadratic order in the dreibein fields,

$$\sqrt{g} = 1 - \frac{1}{2}h^\mu_\mu + \frac{1}{4}\left(\frac{1}{2}h^\mu_\mu h^\nu_\nu - h^\mu_\nu h^\nu_\mu\right) + \ldots \tag{4.23}$$

where

$$g_{\mu\nu} = \eta_{\mu\nu} + e^a_\mu e^a_\nu - \eta_{\mu a}e^a_\nu - \eta_{\nu a}e^a_\mu \equiv \eta_{\mu\nu} + h_{\mu\nu} \tag{4.24}$$

is the expansion of the dynamical metric field about the flat background. Substituting these expansions into the action (4.21) we find

$$\begin{aligned}
S^{(1)}_m[A,\phi,e] \;=\; \int d^3x\, \Bigg\{ & -\frac{1}{4e^2}F^2 + |(\partial - iA)\phi|^2 - m^2|\phi|^2 + \frac{k}{8\pi}AdA \\
& - \frac{1}{4e^2}\Big(\eta^{\mu\lambda}\eta^{\nu\rho}e^\alpha_\alpha - 2\eta^{\mu\lambda}\eta^{\rho a}e^\nu_a - 2\eta^{\nu\rho}\eta^{\lambda a}e^\mu_a - \frac{3}{2}\eta^{\mu\lambda}\eta^{\nu\rho}e^\alpha_\sigma e^\sigma_\alpha + \eta^{\mu\lambda}e^\nu_a e^\rho_a \\
& + \eta^{\nu\rho}e^\mu_a e^\lambda_a + 4\eta^{\lambda a}\eta^{\rho b}e^\mu_a e^\nu_b - 2\eta^{\mu\lambda}\eta^{\rho a}e^\nu_a e^\alpha_\alpha - 2\eta^{\nu\rho}\eta^{\lambda a}e^\mu_a e^\alpha_\alpha \\
& + \frac{1}{2}\eta^{\mu\lambda}\eta^{\nu\rho}e^\alpha_\alpha e^\sigma_\sigma\Big)F_{\mu\nu}F_{\lambda\rho} + \Big(\eta^{\mu\nu}e^\lambda_\lambda + 2\eta^{\nu a}e^\mu_a - \eta^{\mu\nu}e^\rho_\lambda e^\lambda_\rho - e^\mu_a e^\nu_a \\
& + 2\eta^{\nu a}e^\mu_a e^\lambda_\lambda + \frac{1}{2}\eta^{\mu\nu}e^\lambda_\lambda e^\rho_\rho\Big)[(\partial_\mu - iA_\mu)\phi]^*[(\partial_\nu - iA_\nu)\phi] \\
& - \frac{m^2}{4}\left(2e^\mu_\mu + e^\mu_\mu e^\nu_\nu - 3e^\mu_\nu e^\nu_\mu\right)\phi^*\phi \Bigg\}
\end{aligned} \tag{4.25}$$

From (4.25) we can write down the Feynman rules associated with each of the graviton-matter interactions relevant for the total one-loop amplitude. The usual electrodynamical interactions and propagators are as in the last Section. The graviton-photon-photon vertex is

$$\mathcal{F}^\mu_{i\nu\rho}(p;q,r) = \frac{i}{2e^2}\left[(r\cdot p)\delta^\mu_i\eta_{\nu\rho} - \delta^\mu_i r_\nu p_\rho + p^\mu r_\nu \eta_{i\rho} + 2r_i p_\rho \delta^\mu_\nu - 2(r\cdot p)\delta^\mu_\nu\eta_{i\rho} - 2p^\mu r_i \eta_{\nu\rho}\right] \tag{4.26}$$

where $p = q + r$, while the graviton-graviton-photon-photon vertex function is given by

$$\begin{aligned}
\mathcal{F}^{\mu\nu}_{ij\lambda\rho}(p,s;q,r) \;=\; \frac{i}{e^2}\Big[ & 2p_i r_j \delta^\mu_\lambda\delta^\nu_\rho - 2p_i r^\nu \delta^\mu_\lambda \eta_{j\rho} - (p\cdot r)\delta^\mu_\lambda\delta^\nu_\rho\eta_{ij} + p_\rho r^\nu \delta^\mu_\lambda\eta_{ij} + p^\mu r_\lambda \delta^\nu_\rho \eta_{ij} \\
& - p^\mu r^\nu \eta_{ij}\eta_{\lambda\rho} - \frac{1}{2}(p\cdot r)\eta_{\lambda\rho}\delta^\mu_i\delta^\nu_j + \frac{1}{2}p_\rho r_\lambda \delta^\mu_i\delta^\nu_j \\
& + \frac{1}{4}(p\cdot r)\eta_{\lambda\rho}\eta^{\mu\nu}\eta_{ij} - \frac{1}{4}p_\rho r_\lambda \eta^{\mu\nu}\eta_{ij}\Big]
\end{aligned} \tag{4.27}$$

with $p+s=q+r$. The meson-meson-graviton vertex is

$$\mathcal{E}_i^\mu(p,p';q)=i(\delta_i^\mu[p\cdot p'-m^2/2]+p'^\mu p_i+p^\mu p'_i) \tag{4.28}$$

where $q=p-p'$, the meson-meson-graviton-graviton vertex function is

$$\mathcal{E}_{ij}^{\mu\nu}(p,p';q,r)=i\left[\frac{m^2}{4}\left(3\delta_i^\nu\delta_j^\mu-\delta_i^\mu\delta_j^\nu\right)+\frac{1}{2}p\cdot p'\left(\delta_i^\mu\delta_j^\nu-2\delta_j^\mu\delta_i^\nu\right)-p'^\mu p^\nu\eta_{ij}+\delta_j^\nu\left(p_ip'^\mu+p'_ip^\mu\right)\right] \tag{4.29}$$

with $q+r=p-p'$, and the meson-meson-photon-graviton vertex is

$$\mathcal{E}_{i\nu}^\mu(p,p';q;r)=-i[\delta_i^\mu(p+p')_\nu+\delta_\nu^\mu(p+p')_i+\eta_{i\nu}(p+p')^\mu] \tag{4.30}$$

where $q+r=p-p'$. Finally, the meson-meson-photon-graviton-graviton vertex is given by

$$\mathcal{S}_{\mu ij}^{\nu\lambda}(p,p';q;r,s)=-i[\delta_j^\lambda\{(p+p')_i\delta_\mu^\nu+\eta_{\mu i}(p+p')^\nu\}-\delta_j^\nu\delta_i^\lambda(p+p')_\mu-\eta_{ij}(\delta_\mu^\nu p^\lambda+\delta_\mu^\lambda p'^\nu)] \tag{4.31}$$

where $q+r+s=p-p'$. For the one-loop gravitational corrections to the tree-level Aharonov-Bohm amplitude in Fig. 3.1, we can ignore the meson interactions involving two photon lines.

## 4.1 Ward-Takahashi Identities in the Presence of Gravitational Dressing

Before carrying through with the calculation of the required Feynman diagrams, we shall discuss a bit the simplifications and cancellations of certain graphs which arise from the $U(1)$ gauge invariance of the gravitationally dressed Maxwell-Chern-Simons theory above. As we saw in Section 2, the standard set of Slavnov-Taylor identities (or Ward identities in the abelian case) simplified the calculation of the full one-loop order amplitude by reducing the calculation of some combinations of diagrams to a single integration (or cancelling them out in the case $G=U(1)$). The point we wish to make here is that the usual Ward-Takahashi identities associated with the $U(1)$ symmetry of quantum electrodynamics through its minimal coupling to a conserved matter current are still valid when coupled to gravity as above. After integrating the Maxwell $F^2$ term in (4.21) by parts, the current to which the gauge field $A$ is coupled in the gravitationally dressed theory is

$$\mathcal{J}^\mu(x)=\frac{\delta S_m^{(\mathrm{int})}}{\delta A_\mu}=J^\mu(x)+\mathcal{G}^\mu(x) \tag{4.32}$$

where

$$J^\mu=i\sqrt{g}\;g^{\mu\nu}[\phi^*(\partial_\nu-iA_\nu)\phi-\phi((\partial_\nu-iA_\nu)\phi)^*] \tag{4.33}$$

is the usual meson current, and

$$\mathcal{G}^\mu = \frac{1}{2}\sqrt{g}\ g^{\mu\rho}g^{\lambda\nu}\mathcal{P}_{\rho\lambda}A_\nu \tag{4.34}$$

is the current from the gravitational interaction. Here

$$\mathcal{P}_{\mu\nu} = g_{\mu\nu}\nabla^2 - \nabla_\mu\nabla_\nu \tag{4.35}$$

is the symmetric, covariant transverse projection operator on the space of vectors in configuration space and $\nabla^2 = g^{\mu\nu}\partial_\mu(\sqrt{g}\partial_\nu)/\sqrt{g} = g^{\mu\nu}\nabla_\mu\nabla_\nu$ the scalar Laplacian associated with the spacetime metric $g$. In the covariant gauge $\eta^{\mu\nu}\partial_\mu A_\nu = 0$, the current conservation law $\partial_\mu\mathcal{J}^\mu = 0$ follows from taking a derivative of the usual Euler-Lagrange equations

$$0 = \frac{\delta S_m}{\delta A_\mu} = \partial_\nu\left(\sqrt{g}\ g^{\mu\lambda}g^{\nu\rho}F_{\lambda\rho}\right) - \frac{k}{4\pi}\epsilon^{\mu\nu\lambda}F_{\nu\lambda} + J^\mu \tag{4.36}$$

for the photon field, and using the Bianchi identity $\partial_\mu\epsilon^{\mu\nu\lambda}F_{\nu\lambda} = 0$. Moreover, it is readily verified that the gravitational current (4.34) is conserved, $\partial_\mu\mathcal{G}^\mu = 0$, which follows from symmetry and transversality of the projection operator (4.35).

Thus, the meson and gravitational currents which couple to the photon field are individually conserved. This will lead to a set of Ward-Takahashi identities for gravitationally corrected Green's functions in the theory. The first property we wish to establish is the transversality of the gravitationally corrected photon propagator. Consider the function

$$\bar{G}_{\mu\nu}(p) = G_{\mu\nu}(p) + G_{\mu\lambda}(p)\langle J^\lambda(p)A_\nu(-p)\rangle + G_{\mu\lambda}(p)\langle\mathcal{G}^\lambda(p)A_\nu(-p)\rangle \tag{4.37}$$

where we have Fourier transformed the above operators to momentum space. Here $G_{\mu\nu}(p)$ is the free photon propagator (3.2) and the vacuum correlation functions correspond to time-ordered Green's functions in the full interacting theory (4.21). After shifting the graviton field as in (4.7), the gravitational current decomposes as

$$\mathcal{G}^\mu = \mathcal{G}^{(0)\mu} + \mathcal{G}^{(h)\mu} \tag{4.38}$$

into the sum of a piece $\mathcal{G}^{(h)\mu}$ associated with the fluctuation metrics $h_{\mu\nu}$ in (4.24) and a "flat" part

$$\mathcal{G}^{(0)\mu} = \frac{1}{2}\eta^{\mu\rho}\eta^{\lambda\nu}\mathcal{P}^{(0)}_{\rho\lambda}A_\nu \tag{4.39}$$

where $\mathcal{P}^{(0)}_{\mu\nu} = \eta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu$ is the transverse projection operator on the Minkowski space of vectors. In momentum space, this projection operator is the usual one $\mathcal{P}^{(0)}_{\mu\nu}(p) = p^2\eta^\perp_{\mu\nu}(p)$, so that the flat current (4.39) is just the transverse part of the gauge field in Minkowski space. The meson current $J^\mu$ also decomposes into a "flat" part associated with the pure electrodynamical meson-photon interactions and a piece corresponding to meson-photon-graviton

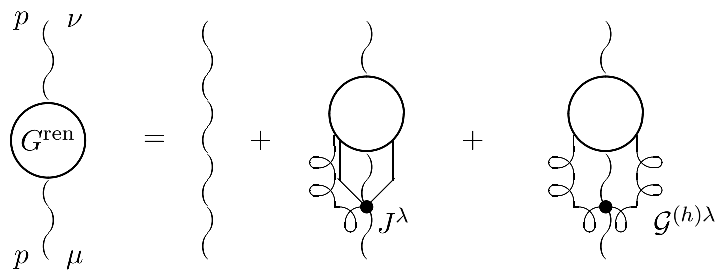



**Figure 4.1:** The complete photon propagator $G^{\rm ren}_{\mu\nu}(p)$ in the gravitationally-dressed topologically massive gauge theory. The spiral lines denote the dreibein fields $e^a_\mu$, and, as before, wavy lines depict the photon fields and straight lines the scalar fields. Here the external legs of the Feynman diagrams are dressed with free propagators $G_{\mu\nu}(p)$, that on the right-hand side of the equality are contracted into insertions of the specified currents which are denoted by the solid circles. The first set of propagators on the right-hand side generated by $J^\lambda$ contain all initial, bottom-most vertices including both the pure electromagnetic meson-photon 3- and 4-point interactions as well as their (infinitely-many) couplings to the graviton field. In the second set only the (infinitely-many) photon-graviton couplings inserted by $\mathcal{G}^{(h)\lambda}$ appear in the bottom-most interaction vertices of the diagrams.

vertices from the $h_{\mu\nu}$ part. By definition, the full renormalized photon propagator in the interacting quantum field theory (4.21) can be written as (see Fig. 4.1)

$$G^{\rm ren}_{\mu\nu}(p) \equiv \langle A_\mu(p)A_\nu(-p)\rangle = G_{\mu\nu}(p) + G_{\mu\lambda}(p)\langle[J^\lambda(p)+\mathcal{G}^{(h)\lambda}(p)]A_\nu(-p)\rangle \tag{4.40}$$

It includes the free photon propagator, the renormalizations from the scalar-photon vertices, and all the renormalizations from the gravity couplings in (4.21). Then substituting the decomposition (4.38) into (4.37) we get

$$\bar{G}_{\mu\nu}(p) = G^{\rm ren}_{\mu\nu}(p) + p^2 G_{\mu\lambda}(p)\eta^{\lambda\rho}\eta^{\alpha\beta}\eta^{\perp}_{\rho\alpha}(p)G^{\rm ren}_{\beta\nu}(p)/2 \tag{4.41}$$

where the second term in (4.41) projects out the transverse component of the complete photon propagator.

In a covariant gauge $\eta^{\mu\nu}\partial_\mu A_\nu = 0$, the free photon propagator obeys $p^\mu G_{\mu\nu}(p) = \xi(p^2)p_\nu$ ($\xi(p^2)=0$ in the Landau gauge). From the canonical commutation relations of the quantum field theory we have

$$[J^0(x), A_\mu(y)]\delta(x^0-y^0) = [\mathcal{G}^0(x), A_\mu(y)]\delta(x^0-y^0) = 0 \tag{4.42}$$

These commutation relations follow from the fact that the canonical momentum conjugate to the gauge field $A_\mu$ can be taken to be proportional to $\epsilon^{0\mu\nu}A_\nu$ in the Chern-Simons gauge theory. By current conservation we have $p_\mu J^\mu(p) = p_\mu \mathcal{G}^\mu(p) = 0$, and so using (4.42) and the transversality of the free photon propagator we have

$$p^\mu \bar{G}_{\mu\nu}(p) = p^\mu G_{\mu\nu}(p) = p^\mu G^{\rm ren}_{\mu\nu}(p) \tag{4.43}$$

where the first equality in (4.43) follows from the definition (4.37) and the second from (4.41) using transversality of the second term there. This means that the longitudinal parts of $G^{\rm ren}_{\mu\nu}$ and $G_{\mu\nu}$ coincide, and so the longitudinal part of the full photon propagator is not renormalized. This holds for the corrections from the photon, meson and graviton fields. In particular, we can equate the various orders of the perturbative expansions in both coupling constants $1/k$ and $1/k'$ to deduce that this result holds for each of the pure gravitational, pure electrodynamical and mixed gravity-electrodynamic corrections, at each given order of perturbation theory in the gravitational and gauge couplings. Thus the vacuum polarizations from each set of corrections will be individually transverse. Denoting the vacuum polarization due to pure gravitational radiative corrections by $\Pi^{\rm (grav)}_{\mu\nu}(p)$, we therefore have the transverse decomposition

$$\Pi^{\rm (grav)}_{\mu\nu}(p)=\frac{1}{M}\Pi^{\rm (grav)}_{e}(p^2)p^2\eta^{\perp}_{\mu\nu}(p)-\frac{k}{4\pi}\Pi^{\rm (grav)}_{o}(p^2)\epsilon_{\mu\nu\lambda}p^{\lambda} \tag{4.44}$$

into the usual parity even and odd components.

The other gravitationally dressed Ward-Takahashi identity we wish to point out is the usual relation between the meson-meson-photon vertex function and the meson self-energy. Consider the full renormalized vertex function

$$V^{\rm ren}_{\mu}(p,q)=\langle A_\mu(q)\phi^*(p-q)\phi(p)\rangle \tag{4.45}$$

and define

$$\Xi^\mu(p,q)=\langle J^\mu(q)\phi^*(p-q)\phi(p)\rangle+\langle\mathcal{G}^\mu(q)\phi^*(p-q)\phi(p)\rangle\equiv\Xi^\mu_\phi(p,q)+\Xi^\mu_g(p,q) \tag{4.46}$$

Expanding the metric about a flat background as before and noting that by definition we have (see Fig. 4.2)

$$G_{\mu\nu}(q)\langle[J^\nu(q)+\mathcal{G}^{(h)\nu}(q)]\phi^*(p-q)\phi(q)\rangle=V^{\rm ren}_\mu(p,q) \tag{4.47}$$

it follows that

$$G_{\mu\nu}(q)\ \Xi^\nu(p,q)=V^{\rm ren}_\mu(p,q)+q^2G_{\mu\nu}(q)\eta^{\nu\rho}\eta^{\lambda\alpha}\eta^{\perp}_{\rho\lambda}(q)V^{\rm ren}_\alpha(p,q)/2 \tag{4.48}$$

where the second term in (4.48) projects out the transverse component of the vector function $V^{\rm ren}_\mu$.

From the canonical commutation relations of the quantum field theory we have

$$[J^0(x),\phi(y)]\delta(x^0-y^0)=-\phi(x)\delta^{(3)}(x-y)\quad,\quad[J^0(x),\phi^*(y)]\delta(x^0-y^0)=\phi^*(x)\delta^{(3)}(x-y)$$
$$[\mathcal{G}^0(x),\phi(y)]\delta(x^0-y^0)=[\mathcal{G}^0(x),\phi^*(y)]\delta(x^0-y^0)=0 \tag{4.49}$$

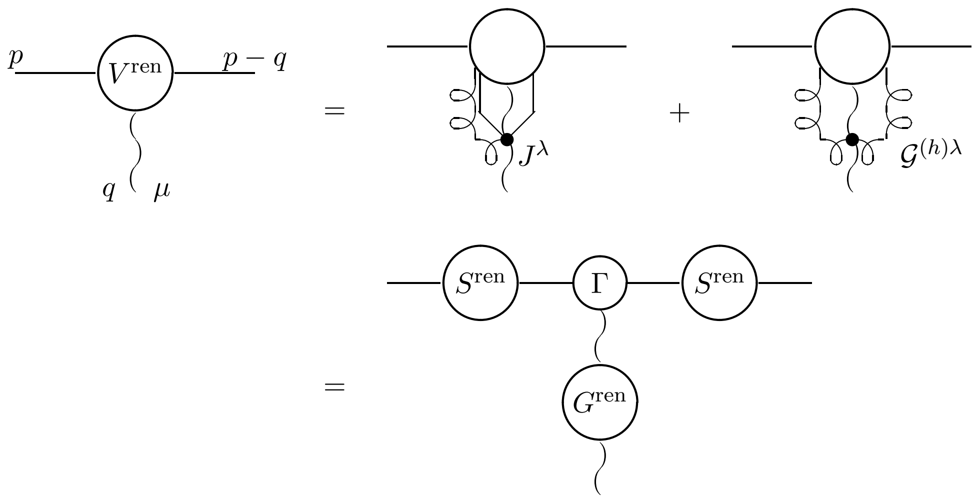



**Figure 4.2:** The complete meson-meson-photon vertex function $V_\mu^{\rm ren}(p,q)$ in the gravitationally-dressed topologically massive gauge theory. The first equality depicted is as in Fig. 4.1 above. The second equality is the usual decomposition of the vertex function into the proper vertex function $\Gamma_\lambda(p,q)$ dressed with complete propagators.

and along with current conservation these lead to the identities

$$q_\mu ~\Xi_\phi^\mu(p,q) = S^{\rm ren}(p-q) - S^{\rm ren}(p) \qquad , \qquad q_\mu ~\Xi_g^\mu(p,q) = 0 \tag{4.50}$$

where $S^{\rm ren}(p) = \langle\phi^*(p)\phi(-p)\rangle$ is the full renormalized meson propagator. In a transverse gauge where $q^\mu G_{\mu\nu}(q) = \xi(q^2)q_\nu$, the contraction of the equation (4.48) with $q^\mu$ yields

$$q^\mu G_{\mu\nu}(q) ~\Xi^\nu(p,q) = q^\mu V_\mu^{\rm ren}(p,q) = \xi(q^2)[S^{\rm ren}(p-q) - S^{\rm ren}(p)] \tag{4.51}$$

where the first equality in (4.51) follows from transversality of the projection operator in the second term in (4.48), while the second equality follows from (4.50).

Now notice that the usual Furry theorem of quantum electrodynamics holds here, because the quantum field theory (4.21) is invariant under the charge conjugation transformation

$$(\phi, A_\mu, e_\mu^a) \xrightarrow{C} (\phi^*, -A_\mu, e_\mu^a) \tag{4.52}$$

This implies that the only non-vanishing Green's functions of the quantum field theory are those that contain an even number of external photon lines and the same number of incoming $\phi$ and outgoing $\phi^*$ external lines. It does not, however, restrict graviton lines. We can therefore define the one-particle irreducible vertex function $\Gamma_\mu$ by the identity (Fig. 4.2)

$$V_\mu^{\rm ren}(p,q) = \eta^{\lambda\nu} G_{\mu\lambda}^{\rm ren}(q) S^{\rm ren}(p-q) i\Gamma_\nu(p,q) S^{\rm ren}(p) \tag{4.53}$$

Contracting the equation (4.53) with $q^\mu$ and using tranversality of the free photon propagator in (4.43) and the identity (4.51) leads to the usual $U(1)$ Ward-Takahashi identity

$$q^\mu \Gamma_\mu(p,q) = \Sigma(p-q) - \Sigma(p) \tag{4.54}$$

between the longitudinal part of the irreducible vertex function and the one-particle irreducible meson self-energy defined by $S^{\rm ren}(p)^{-1} = S(p)^{-1} + \Sigma(p)$. The relation (4.54) holds to all orders of $1/k$ and $1/k'$ in the electrodynamical and gravitational perturbative expansions. In particular, it applies to the pure gravitational corrections as discussed above.

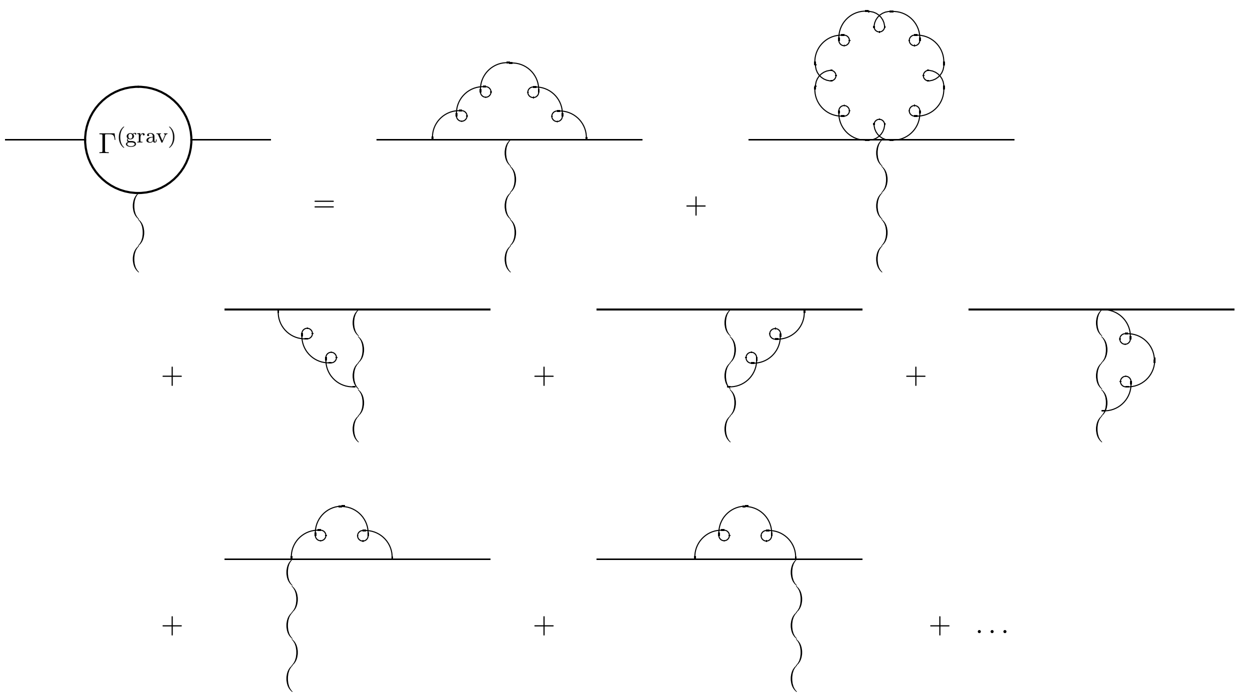
## 4.2 Gravitational Contributions to the Vacuum Polarization

We shall now begin computing the radiative corrections due to gravity to the Aharonov-Bohm scattering of two charged mesons depicted in Fig. 3.1. We note first of all that the sum of the amplitudes depicted in Figs. 3.3 and 3.4 vanishes, where the one-loop graviton contributions to the irreducible vertex function and the meson self-energy are shown in Fig. 4.3. This follows from the Ward-Takahashi identity (4.54) which cancels the longitudinal contribution of the vertex function with the self-energy contributions (c.f. Fig. 3.5). The remaining transverse contribution from the vertex function (which combines with the parity-even part of the photon propagator) is easily checked to be non-singular in the limit of zero momentum transfer, and so it does not contribute to the Aharonov-Bohm amplitude. This non-singular behaviour of renormalized quantities is expected on general grounds because of the infrared finiteness of the Yang-Mills theory of Section 3 in the Landau gauge [37, 38]. These simplifications cancel out 22 potential contributions.

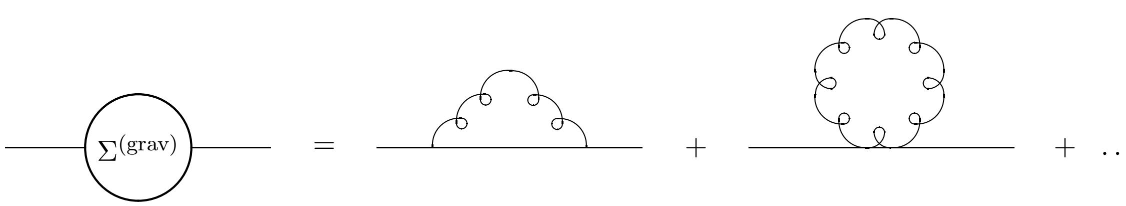
We next consider the gravitational corrections to the photon propagator at one-loop order (Fig. 4.4). The gravitational vacuum polarization tensor is given by

$$\begin{aligned}\Pi^{(\rm grav)}_{\mu\nu}(q) = \int \frac{d^3k}{(2\pi)^3}\ \Big\{&\eta^{\lambda\sigma}\eta^{\rho\kappa}\mathcal{F}^{\alpha}_{i\mu\sigma}(k;q,k-q)\mathcal{F}^{\beta}_{j\kappa\nu}(-k;q-k,-q)G_{\lambda\rho}(q-k)D^{ij}_{\alpha\beta}(k)\\ &+\mathcal{F}^{\alpha\beta}_{ij\mu\nu}(k,-k;q,-q)D^{ij}_{\alpha\beta}(k)\Big\}\end{aligned} \tag{4.55}$$

This integral is ultraviolet finite and can be evaluated using dimensional regularization [41]. As in the Yang-Mills case, the tadpole contribution (second term in (4.55)) doesn't contribute to the required parity-odd structure. Only the parity odd part of (4.55) contributes to the Aharonov-Bohm part of the amplitude shown in Fig. 3.2, and it comes from either the $\epsilon$-term in the photon propagator or the $\epsilon$-terms in the graviton propagator. Contracting (4.55) with




(a)




(b)

**Figure 4.3:** Total one-loop gravitational renormalization of (a) the meson-meson-photon vertex, and (b) the meson propagator.

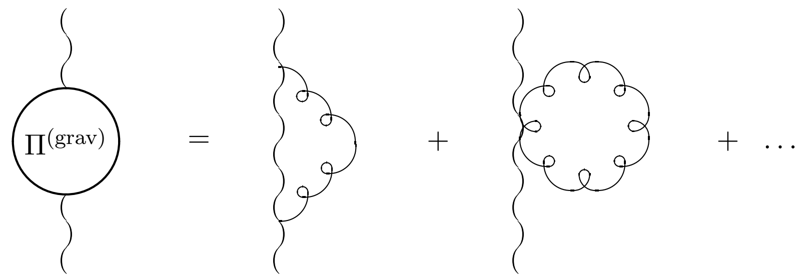



**Figure 4.4:** Total one-loop gravitational renormalization of the photon propagator.

$\frac{k}{4\pi}\frac{\epsilon_{\mu\nu\lambda}q^\lambda}{2q^2}$, after some lengthy algebra this piece is found to be

$$\begin{aligned}\Pi_o^{(\mathrm{grav})}(q^2) \;=\; & \frac{\pi e^2}{2k'\mu}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{k^2(k^2-\mu^2)[(k+q)^2-M^2]}\left\{(2M+\mu/2)q^4-\mu\frac{q^6}{2k^2}\right.\\ & +(2M+5\mu/2)(k+q)^4-4Mq^2(k+q)^2\\ & +\mu(k+q)^2\left[\frac{(k+q)^4}{2k^2}-3q^2-\frac{q^2(k+q)^2}{k^2}+\frac{q^4}{k^2}\right]\\ & \left.+\frac{1}{2}\mu k^2\left[q^2-5(k+q)^2\right]-(2M+\mu/2)k^4\right\}\end{aligned} \tag{4.56}$$

The evaluation of this integral has been discussed in [41] where it was shown to vanish as $q \to 0$ in dimensional regularization. Consequently, there is no gravitational renormalization of the Chern-Simons gauge coefficient $k$ and gravitational radiative corrections do not dynamically generate topological vector boson mass terms at one-loop order. Thus, in contrast with the pure gauge theory renormalization, the gravitational contributions to the vacuum polarization do not affect the Aharonov-Bohm interaction, and hence the gravitationally dressed conformal dimensions, at one-loop order.

## 4.3 Graviton Ladder Exchange Contributions

We see therefore that the one-loop gravitational renormalization of the Aharonov-Bohm interaction receives contributions from *only* the ladder diagrams shown in Fig. 4.5. Note that we have depicted only the topologically inequivalent Feynman diagrams here. There are actually 9 ladder graphs in total. The "triangle" diagram in Fig. 4.5 has a counterpart corresponding to $p_1 \leftrightarrow p_2$, $q \to -q$ and the "box" graph has a partner corresponding to $p_2 \to -(p_2+q)$ (see Fig. 3.6). Also, with the exception of the last "circle" diagram, each diagram has a partner associated with interchanging the photon and graviton lines (equivalently the interchange of initial and final particles in the diagrams). The situation we observe here is opposite to that in the case of the topologically massive Yang-Mills theory of the last Section, where the ladder

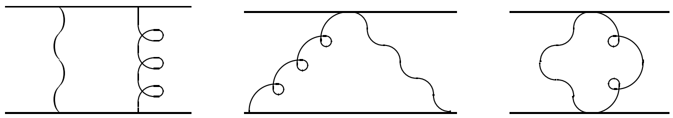


**Figure 4.5:** The topologically inequivalent one-loop gravitational ladder (1-photon and 1-graviton exchange) diagrams.

diagrams vanish in the anyon limit and the conformal weight renormalization is determined entirely by the gluon loop self-energy and the gluon renormalization of the ghost propagator. Thus the perturbative, infrared structure of topologically massive gravity is in this sense opposite to that of topologically massive gauge theory.

Let us remark at this stage that the simplification of the amplitude to the evaluation of only the ladder graphs occurs as well in the non-abelian case. Although the pure electrodynamical Ward-Takahashi identities become the more complicated Slavnov-Taylor identities discussed in the previous Section, those identities associated with the gravitational corrections are still valid because the gravitational current $\mathcal{G}^\mu$ is once again given by a (non-abelian) projection operator. Furthermore, the ghost fields associated with the Yang-Mills gauge fields do not contribute to the gravitational corrections because the gauge-fixing and ghost terms in the action are decoupled from the graviton field $e_\mu^a$. The only difference in the non-abelian case is that the bare conformal dimension $\Delta_0$ of the associated primary operators changes.

We are interested in the parity odd parts of the ladder diagrams, i.e. the integrand terms which can contribute to the $\epsilon_{\mu\nu\lambda}p_1^\mu p_2^\nu q^\lambda$ structure of the full amplitude relevant for the gravitational renormalization of the Aharonov-Bohm interaction. This structure comes from contracting the parity-even part of the photon propagator with the parity odd part of the graviton propagator, and vice versa. As in the pure gauge theory case, the parity even parts of the diagrams will represent a renormalization of the Coulomb charge interaction between the charged mesons (which vanishes in the anyon limit) and of the Pauli magnetic moment interaction (which is finite in the anyon limit) [29]. To simplify the integrands of the corresponding Feynman integrals, we use the on-shell conditions for the initial and final scalar particles, i.e. $p_1^2 = (p_1 - q)^2 = m^2$ and $p_2^2 = (p_2 + q)^2 = m^2$. We can also exploit the tranversality in the Landau gauge of the free photon propagator $G_{\mu\nu}(p)$ in both of its indices, and of the free graviton propagator $D_{\mu\nu}^{ij}(p)$ in all four of its indices. After some very lengthy and tedious algebra, the Feynman integrals for the parity-odd parts of the ladder amplitudes depicted in Fig. 4.5 can be simplified to the following expressions,

$$
\begin{aligned}
L_{\Box} &= -i\int \frac{d^3k}{(2\pi)^3}\,(2p_1-q-k)^\mu(2p_2+q-k)^\nu \mathcal{E}_i^\lambda(k+p_1-q,p_1-q)\mathcal{E}_j^\rho(p_2+q-k,p_2+q) \\
&\qquad\qquad \times G_{\mu\nu}(q-k)D^{ij}_{\lambda\rho}(k)S(k+p_1-q)S(p_2+q-k) \\
&= -\frac{16e^2M}{\kappa}\int\frac{d^3k}{(2\pi)^3}\,\frac{\epsilon_{\mu\nu\lambda}p_1^\mu p_2^\nu (q-k)^\lambda}{k^2(k^2-\mu^2)(k-q)^2[(k-q)^2-M^2][k^2+2k\cdot(p_1-q)]} \\
&\qquad \times \frac{1}{k^2-2k\cdot(p_2+q)}\left\{(k^2+\mu^2)\left[\frac{3m^2}{2}\left(\frac{5m^2}{6}+k\cdot(p_1-p_2-2q)\right)\right.\right. \\
&\qquad \left.+\frac{[k\cdot(p_2+q)]^2}{k^2}\left(k\cdot(q-p_1)-\frac{m^2}{2}\right)+\frac{[k\cdot(p_1-q)]^2}{k^2}\left(k\cdot(p_2+q)-\frac{m^2}{2}\right)\right] \\
&\qquad +2\mu^2\left(m^2-\frac{[k\cdot(p_1-q)]^2}{k^2}\right)\left(m^2-\frac{[k\cdot(p_2+q)]^2}{k^2}\right) \\
&\qquad \left.+2(k^2-\mu^2)\left[(p_1-q)\cdot(p_2+q)-\frac{[k\cdot(p_1-q)][k\cdot(p_2+q)]}{k^2}\right]^2\right\}
\end{aligned}
\tag{4.57}
$$

$$
\begin{aligned}
L_{\triangle} &= \mathcal{E}_i^{\mu\lambda}(p_1,p_1-q)\int\frac{d^3k}{(2\pi)^3}\,(2p_2+q+k)^\nu\mathcal{E}_j^\rho(p_2,p_2+k)G_{\mu\nu}(q-k)D^{ij}_{\lambda\rho}(k)S(k+p_2) \\
&= -\frac{2e^2M}{\kappa}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{k^2(k^2-\mu^2)(k-q)^2[(k-q)^2-M^2](k^2+2k\cdot p_2)} \\
&\qquad \times\left\{\epsilon_{\mu\nu\lambda}p_1^\mu p_2^\nu q^\lambda\left[4m^2(k^2+2m^2)+4(p_2\cdot k)(k^2+\mu^2)-\frac{(p_2\cdot k)^2}{k^2}\left(2k^2+6\mu^2\right)\right]\right. \\
&\qquad +\epsilon_{\mu\nu\lambda}p_2^\nu q^\lambda k^\mu\frac{(p_1\cdot k)}{k^2}\left[2(p_2\cdot k)(3\mu^2-5k^2)-m^2(k^2-5\mu^2)+\frac{4\mu^2}{k^2}(p_2\cdot k)^2\right] \\
&\qquad +\epsilon_{\mu\nu\lambda}\left(p_1^\mu p_2^\nu k^\lambda-p_1^\mu q^\lambda k^\nu\right)\left[2(p_2\cdot k)^2-4m^2(k^2+2\mu^2)\right. \\
&\qquad \left.\left.-4(p_2\cdot k)(k^2+\mu^2)+\frac{2\mu^2}{k^2}(p_2\cdot k)^2\right]\right\}
\end{aligned}
\tag{4.58}
$$

$$
\begin{aligned}
L_{\bigcirc} &= i\mathcal{E}_i^{\mu\nu}(p_1,p_1-q)\mathcal{E}_j^{\rho\lambda}(p_2,p_2+q)\int\frac{d^3k}{(2\pi)^3}\,G_{\nu\lambda}(q-k)D^{ij}_{\mu\rho}(k) \\
&= \frac{2e^2M}{\kappa}\int\frac{d^3k}{(2\pi)^3}\,\frac{1}{k^2(k^2-\mu^2)(k-q)^2[(k-q)^2-M^2]}\left\{\epsilon_{\mu\nu\lambda}p_1^\mu p_2^\nu q^\lambda(5k^2-2\mu^2)\right. \\
&\qquad -3\epsilon_{\mu\nu\lambda}p_1^\mu p_2^\nu k^\lambda(k^2+4\mu^2)+\frac{2}{k^2}(k^2-\mu^2)\epsilon_{\mu\nu\lambda}q^\lambda\left[p_1^\mu k^\nu(p_2\cdot k)+k^\mu p_2^\nu(p_1\cdot k)\right] \\
&\qquad \left.-\frac{18}{k^2}\mu^2\epsilon_{\mu\nu\lambda}q^\lambda\left[p_1^\mu k^\nu(p_2\cdot k)+k^\mu p_2^\nu(p_1\cdot k)\right]\right\}
\end{aligned}
\tag{4.59}
$$

The above expressions for the Aharonov-Bohm amplitudes are exact. One intriguing feature of them is that the integrands which come from contracting the even part of the photon propagator with the odd part of the graviton propagator vanish identically. In fact, we can work out the tree-level gravitational Aharonov-Bohm interaction amplitude (i.e. the imaginary part of the Feynman diagram in Fig. 3.1 with the photon line replaced by a graviton line), and after some algebra we find

$$\mathcal{T}^{(\mathrm{grav})}(p_1,p_2;q)^{\mathrm{odd}} = i\mathcal{E}_i^{\mu}(p_1,p_1-q)\mathcal{E}_j^{\nu}(p_2,p_2+q)D_{\mu\nu}^{ij}(q)^{\mathrm{odd}} \equiv 0 \tag{4.60}$$

where we have included only the parity-odd $\epsilon$-terms of the full graviton propagator (4.20). As (4.60) vanishes identically (for all ranges of momenta), there is no gravitational Aharonov-Bohm effect. The one-loop computations above then show that there is no induced gravitational Aharonov-Bohm effect from the interactions of the topologically massive graviton field with the meson and photon fields, i.e. the photon field does not renormalize the gravitational Chern-Simons term. Thus the only effect of the gravitational dressing in the infrared regime is to change the induced spin of the meson field due to its interaction with the Chern-Simons gauge field. Although this gauge interaction is dressed by gravity, the gravitational interaction is not dressed by the electromagnetic field[5].

As the initial spin of the charged particles here is zero, this result is consistent with the KPZ formula for the branch which has $\Delta(0) = 0$. One can check that this feature is also true of the higher-loop amplitudes involving only graviton lines, which is a result of the index contractions which occur in the integrands of the Feynman integrals. This therefore agrees with the expansion (2.13) of the KPZ formula in which each term is at least of order $1/k$, i.e. there are no terms at any given order of the $1/k'$ expansion which aren't accompanied by factors of $1/k$. It is only for particles with non-zero bare spin, such as fermions or vector bosons, that one would find non-vanishing pure gravitational corrections. In these cases, the non-zero diagrams would presumably come from the coupling of the spinning fields to the spin-connection $\omega_\mu^a$ itself.

In the Appendices some of the technical details of the evaluation of the ladder diagrams (4.57)–(4.59) are discussed. Here we simply quote the final results of their evaluation in the infrared limit. We write each of the ladder amplitudes above as

$$L(p_1,p_2;q) = \frac{\epsilon_{\mu\nu\lambda}p_1^{\mu}p_2^{\nu}q^{\lambda}}{q^2}\,\mathcal{L}(q^2) \tag{4.61}$$

where $\mathcal{L}(q^2) = \sum_n (q^2)^n\ \mathcal{L}^{(n)}$ are Lorentz-invariant functions whose coefficients $\mathcal{L}^{(n)}$ depend only on the bare parameters of the matter-coupled topologically massive gravity theory and

[5]Note that this differs from the gravitational Aharonov-Bohm effect where spinless, non-dynamical point particles in the presence of a topologically massive gravity field acquire an induced spin $k'/32\pi^2$ under adiabatical rotation [39]. This gravitational Aharonov-Bohm effect is non-perturbative and comes from the abelian, linearized approximation to the action.

the kinematical invariants $p_1 \cdot p_2$, $p_1 \cdot q$ and $p_2 \cdot q$. Each of the separate Feynman integrals in (4.57)–(4.59) are found to converge in the infrared limit $e^2, \kappa \to \infty$. Some of these terms produce naively divergent contributions when multiplied by the overall coefficient $e^2 M/\kappa \sim M^2/\mu$, but these terms cancel out when the individual integrals are combined in the full amplitude in the small momentum limit $q \to 0$. This is as hoped for, since the final result for the induced spin should not depend on any mass scale of the model.

The functions $\mathcal{L}(q^2)$ above are found to be analytic around $q^2 = 0$, thus confirming previous arguments about the infrared finiteness of topologically massive gravity in the Landau gauge [37, 38]. A non-zero $\mathcal{L}(q^2 = 0)$ will produce a singular pole term at $q^2 = 0$ in (4.61) and hence a non-vanishing contribution to the Aharonov-Bohm part of the amplitude. After evaluating the Feynman integrals in (4.57)–(4.59) using the techniques sketched in the Appendices, we get the final results for the ladder amplitudes in the infrared limit,

$$
\begin{aligned}
\mathcal{L}_{\square}(0) = \lim_{q^2 \to 0} \; & -\frac{8\pi i}{kk'm^2q^2(p_1 - q)\cdot q(p_2 \cdot q)} \Big[8q^6(p_1 \cdot p_2) + 2q^4\Big(m^2(p_2 - p_1)\cdot q \\
& +(p_1 \cdot p_2)(p_2 - 4p_1)\cdot q\Big) + 2q^2\Big(m^2(p_1 \cdot q)^2 + m^2(p_2 \cdot q)^2 \\
& -3(p_1 \cdot p_2)(p_1 \cdot q)(p_2 \cdot q)\Big) + \pi^2\Big(m^2q^2\Big\{(p_1 \cdot q)(p_2 \cdot q) + 8(p_2 \cdot q)^2 \\
& +2(p_1 \cdot q)^2\Big\} - m^2q^4 p_1 \cdot (2q + p_2) + 2q^4(p_1 \cdot p_2)(3p_2 - 4p_1)\cdot q \\
& - 6m^2(p_1 \cdot q)(p_2 \cdot q)(p_1 + p_2)\cdot q + q^2(p_1 \cdot p_2)(p_2 \cdot q)(3p_2 - 7p_1)\cdot q \\
& +8q^6(p_1 \cdot p_2) - 3(p_1 \cdot p_2)(p_2 \cdot q)(p_1 \cdot q)(p_1 + p_2)\cdot q\Big)\Big] \\
\mathcal{L}_{\triangle}(0) = \lim_{q^2 \to 0} & -\frac{4\pi i}{kk'q^2(p_2 \cdot q)}\Big[6q^4 - 17(p_2 \cdot q)q^2 + \pi^2\Big(16q^4 - 3(p_2 \cdot q)q^2 - 18(p_2 \cdot q)^2\Big)\Big] \\
\mathcal{L}_{\bigcirc}(0) = 0 &
\end{aligned}
\tag{4.62}
$$

We see here that, in contrast to the ladder diagrams of topologically massive gauge theory [30], there is a finite contribution from the graviton ladder exchanges in the infrared limit. That the amplitude $L_{\bigcirc}$ has no leading order contributions to the Aharonov-Bohm amplitude can be understood from the fact that the contraction of the meson-meson-photon-graviton vertices in (4.59) produces the same sort of Feynman integrals that appear in the gauge theory ladder amplitudes (3.37) which individually vanish in the anyon limit. The other ladder amplitudes in (4.57) and (4.58) involve higher-derivative gravitational interaction vertices leading to Feynman integrals of tensorial rank 4 and 5, whereas in the gauge theory case the maximum tensorial rank of the Feynman integrals involved is 3 [30]. These higher rank Feynman integrations are ultimately responsible for the non-vanishing contribution here in the anyon limit (see the Appendices), and thus the full gravitationally induced spin comes

from the higher-order couplings of the meson and photon fields to the graviton field. It is interesting that, unlike the scalar-coupled gauge theory case, the external charged mesons here must be dynamical quantum fields to generate a renormalization of the spin. This is similar to the effect of massive spinor fields coupled to topologically massive Yang-Mills theory where the dynamical spin of the fermions produces a non-zero contribution. Here the dynamical nature is needed to induce a non-zero bare spin for the gravitationally interacting particles.

To determine the corresponding conformal weights, we note that the on-shell conditions for the external meson lines imply that $p_1 \cdot q = -p_2 \cdot q = q^2/2$, and also that $p_1 \cdot p_2 \to m^2$ in the limit $q^2 \to 0$. Comparing with Section 2 we can then identify the induced spin associated with each of the amplitudes above as

$$\Delta_{\square} = -\frac{1}{2kk'}\left(14 + 13\pi^2\right) \quad , \quad \Delta_{\triangle} = \frac{29 + 26\pi^2}{4kk'} \quad , \quad \Delta_{\bigcirc} = 0 \tag{4.63}$$

Finally, there are the contributions from the remaining "box" and "triangle" graphs which can be obtained from the above amplitudes by permutations of the external particle lines. The triangle diagrams corresponding to the interchange of the 2 mesons can be obtained from the above amplitudes by interchanging $p_1 \leftrightarrow p_2$ and reflecting $q \to -q$ in them, and the box diagram corresponding to crossing the photon and graviton lines in that depicted in Fig. 4.5 is given by $-L_{\square}(p_2 \to -(p_2 + q))$. The remaining graphs correspond to the interchange of photon and graviton lines and are given in terms of the amplitudes as $L(p_2 \to -(p_2 + q), p_1 \to -(p_1 - q))$, i.e. by the interchange of initial and final particles with sign factors appropriate to the change in direction of the lines. These operations lead to the same Aharonov-Bohm amplitudes as in (4.62), and hence to the same weights. Taking into account the bare conformal weight from the tree-level diagrams, the total conformal dimension up to one-loop order is thus

$$\Delta^{(1)}_{\rm grav} = \Delta_0 + 4\Delta_{\square} + 4\Delta_{\triangle} + \Delta_{\bigcirc} = 1/k + 1/kk' \tag{4.64}$$

which agrees with the leading orders of the expansion (2.13) of the KPZ formula.

This is the main result of this paper – the total one-loop radiative corrections to the Aharonov-Bohm amplitude in matter-coupled topologically massive gravity coincides in the anyon limit with the leading orders of the large-$k'$ expansion of the KPZ formula. It represents a non-trivial correspondence between observables of the two-dimensional conformal field theory and the those of its topological membrane description. Notice that, in contrast to the gauge theory calculation, the conformal dimension (4.64) does not depend on the sign of $k'$, as expected from both the topological limit of the topologically massive gravity model and the identification of the gravitational Chern-Simons coefficient with the central charge of the $SL(2,\mathbb{R})$ current algebra of Liouville theory. The $(2+1)$-dimensional Einstein action can have either sign and it is only the overall sign of the topologically massive gravity action

relative to that of the matter fields that must be fixed to eliminate potential ghost terms. Thus the quantum field theory defined by (4.2) does not depend on $\mathrm{sgn}(k')$ as it did in the topological gauge theory case.

# 5 Conclusions

In this Paper we have demonstrated how the spectrum of anomalous dimensions in conformal field theories can be calculated perturbatively from their associated topological membrane descriptions. In particular, we have derived the leading orders of the KPZ conformal weight formula from the infrared limit of topologically massive gravity coupled to charged scalar fields with non-trivial anomalous spin. The gauge theory induced spins are essentially determined by the well-known renormalization of the current algebra level of the WZNW model in Chern-Simons gauge theory, while the gravitationally dressed dimensions are determined by the higher-derivative couplings of the anomalous spin of the charged fields to the graviton. The above discussions have demonstrated how a perturbative analysis of the topologically massive gravity theory reproduces several consistent features of the KPZ conformal dimension formula, and how the three-dimensional perturbative description provides several intuitive descriptions of the gravity theory in two-dimensions. The coupling of the topologically massive gravity model to the topologically massive gauge theory has also demonstrated the interplay between gauge and gravity fields in the topological membrane description (see Subsection 4.2 above). This illustrates from a perturbative point of view the relations between world-sheet conformal fields and the geometry of random surfaces in the induced string theory.

It would be interesting to carry out higher-order checks of the KPZ formula to check the next orders of its expansion. For instance, the sum of the amplitudes involving only one graviton line but more than two photon lines should vanish since (2.13) contains no order $1/k^n k'$ terms with $n > 2$. Heuristically, the vanishing of gravitationally-corrected ladder amplitudes with large numbers of photon lines is anticipated because then the insertion of photon propagators "softens" the effects of the higher-derivative couplings to the graviton field and the required Feynman integrations begin to resemble those of the pure gauge theory. However, to explicitly check these other aspects of this relation requires evaluation of two- and higher-loop integrals, which seem nearly intractable in light of the large degree of complexity already involved at one-loop order. Further properties and observables of the WZNW and Liouville models could be readily verified at one-loop order.

The calculations presented in this Paper are relevant to the topological membrane description of string theory. It would be interesting to exploit this description as a starting point for (world-sheet) modifications of string theory, especially in light of the recent realization

that the conventional understanding of the (target-space) structure of string theory requires a modification (M-theory). The membrane approach also suggests geometrical interpretations of observables in the two-dimensional theories. For instance, the three-dimensional perturbative calculations produce the gravitationally dressed spin $\Delta$ with the boundary condition $\Delta(\Delta_0 = 0) = 0$. From the point of view of the two-dimensional Liouville theory, there is no immediate reason to choose this branch for the solutions of the KPZ scaling relations. At the calculational level, the topological membrane approach is appealing because it employs standard techniques of quantum field theory, such as perturbative renormalization, to study characteristics of the induced string theory.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## Appendix A Evaluation of the Gravitational Ladder Diagrams

In this Appendix we shall briefly outline the strategy used to evaluate the Feynman integrals (4.57)–(4.59). A partial table of results for the individual Feynman integrals involved can be found in [30] and Appendix B to follow. Here we only discuss the methods used for the sake of completeness. Although the integrations involved in (4.57)–(4.59) can be evaluated using the usual Feynman parameter techniques, they are rather cumbersome and can be simplified to simpler algebraic forms via some manipulations. The exact expressions for many of the individual Feynman integrals involved, and even their infrared limits, are extremely complicated and not at all informative. We therefore omit the detailed expressions and simply discuss the simplifying techniques which were used to arrive at the small momentum results in (4.62). These methods could serve of use for other perturbative calculations in topologically massive gravity.

As mentioned in Subsection 4.3 above, using the results of [30] and carrying out explicit integrations one can see that most of the surviving contributions in the anyon limit $M, \mu \to \infty$ come from the higher derivative contributions in (4.57) and (4.58). We begin by illustrating

why this is so. For brevity, we introduce the following shorthand notations for the propagators appearing in (4.57)–(4.59),

$$D_g(q,M) = (k-q)^2 - M^2 \qquad , \qquad D_m(p) = k^2 + 2k\cdot p \tag{A.1}$$

After shifting $p_1 \to p_1 - q$, $p_2 \to -(p_2+q)$ in (4.57), in the fourth line of the integrand of the second equality there the term

$$\mathcal{B} = \frac{(k\cdot p_1)^2}{k^2}\frac{(k\cdot p_2)^2}{k^2} \tag{A.2}$$

leads to a rank-5 tensor Feynman integral, i.e. one with five loop momentum factors $k^{\mu_1}\cdots k^{\mu_5}$ in the numerator of (4.57). However, most of these tensor components appear contracted with the external meson momenta, and the tensorial rank can be reduced by the trivial identity

$$2k\cdot p = D_m(p) - k^2 \tag{A.3}$$

which at the same time cancels out some of the denominator factors in (4.57). The cancellation of denominator factors is desired so that a minimal number of Feynman parameters will be required at the end for the explicit integrations.

After successive iterations of (A.3) one is left with an algebraically longer set of Feynman integrals to carry out. There are 3 factors of $k^2$ that can be cancelled out from this process, and then any remaining $k^2$ factors in the numerator can be cancelled with denominator factors using

$$k^2 = D_g(0,\mu) + \mu^2 \tag{A.4}$$

It is from here that most of the contributions arise, because the trivial identity (A.4) produces a Feynman integral that is multiplied by an extra overall factor of $\mu^2$. In the anyon limit, this can therefore lead to potential contributions which would otherwise be suppressed by factors of $1/\mu^2$. Such tensorial rank reductions can be used for all of the higher-rank Feynman integrations appearing in the ladder amplitudes. The resulting algebra is somewhat more lengthy, but this is traded off in the simplification that now only Feynman integrals up to tensorial rank 3 which have fewer numbers of denominator factors need be evaluated.

Again some of the Feynman integrations are still rather complicated because of large numbers of denominator factors which appear in them. Some of them can be reduced using the partial fraction decompositions

$$\begin{aligned}(k-q)^{-2}D_g(q,M)^{-1} &= [D_g(q,M)^{-1} - D_g(q,0)^{-1}]/M^2 \\ k^{-2}D_g(0,\mu)^{-1} &= [D_g(0,\mu)^{-1} - D_g(0,0)^{-1}]/\mu^2\end{aligned} \tag{A.5}$$

This reduces the Feynman integrations to those with only the denominator factors $D_g(q,M)$ or $D_g(0,\mu)$, and then the remaining part of the integrations with the factors in (A.5) are

obtained by computing the $M \to 0$ or $\mu \to 0$ limits of these resulting integrals. The partial fraction decomposition (A.5) and any shifts in the loop momentum $k$ are justified because the $(2+1)$-dimensional Feynman integrations involved are absolutely convergent. After using the decompositions (A.5) it is possible to identify after some manipulations whether or not the Feynman integral will contribute in the anyon limit after identifying the overall power of the photon and graviton masses in the expressions for the amplitudes. This can be done by using explicit Feynman parametrizations and asymptotic expansions of the integrands for $M, \mu \to \infty$.

In this way one is left with a (large) succession of Feynman integrals of no more than rank-3 to evaluate. However, the remaining higher-rank integrations still potentially involve more than three distinct denominator factors and therefore require more than two Feynman parameters for their evaluation. They can be simplified and effectively reduced to scalar rank integrals using the elegant Brown-Feynman method which was exploited in the case of one-loop radiative corrections in three-dimensional electrodynamics in [30]. Let us briefly sketch the idea behind this method. As an example, consider the rank-3 tensor Feynman integral

$$J_1^{\mu\nu\lambda} = \int \frac{d^3k}{(2\pi)^3} \, \frac{k^\mu k^\nu k^\lambda}{(k^2)^2 D_g(0,\mu)(q-k)^2 D_g(q,M) D_m(p_2)} \tag{A.6}$$

which arises in the evaluation of the ladder amplitudes (4.57) and (4.58). Using (A.5) we can reduce this to the evaluation of

$$J_2^{\mu\nu\lambda} = \int \frac{d^3k}{(2\pi)^3} \, \frac{k^\mu k^\nu k^\lambda}{k^2 D_g(0,\mu)(q-k)^2 D_g(q,M) D_m(p_2)} \tag{A.7}$$

so that $J_1^{\mu\nu\lambda} = [J_2^{\mu\nu\lambda} - J_2^{\mu\nu\lambda}(\mu = 0)]/\mu^2$. The general structure of the integral (A.7) will be of the form

$$J_2^{\mu\nu\lambda} = a^{\mu\nu} p_2^\lambda + b^{\mu\nu} q^\lambda + c^\mu s^{\nu\lambda} + c^\nu s^{\mu\lambda} \tag{A.8}$$

where $a^{\mu\nu}$, $b^{\mu\nu}$ are tensor-valued functions and $c^\mu$ a vector-valued function of $p_2$ and $q$. The symmetric tensor $s^{\mu\nu}$ is chosen to project out components of vectors transverse to both $p_2$ and $q$, i.e. $p_{2,\mu} s^{\mu\nu} = q_\mu s^{\mu\nu} = 0$, with the normalization $s^\mu_\mu = 1$. Solving these constraints leads to the explicit form

$$s^{\mu\nu} = \eta^{\mu\nu} - \frac{1}{m^2 q^2 - (p_2 \cdot q)^2} \left[ m^2 q^\mu q^\nu + q^2 p_2^\mu p_2^\nu + (p_2 \cdot q)\left( q^\mu p_2^\nu + p_2^\mu q^\nu \right) \right] \tag{A.9}$$

To determine the as yet unknown functions $a^{\mu\nu}$, $b^{\mu\nu}$ and $c^\mu$ above, we first contract both sides of the decomposition (A.8) with $p_2^\mu$ and $q^\mu$ to get

$$2p_{2,\lambda} J_2^{\mu\nu\lambda} = 2m^2 a^{\mu\nu} + 2(p_2 \cdot q) b^{\mu\nu} \quad , \quad 2q_\lambda J_2^{\mu\nu\lambda} = 2(p_2 \cdot q) a^{\mu\nu} + 2q^2 b^{\mu\nu} \tag{A.10}$$

Inside the integrand of (A.7), we then use the identities (A.3) and

$$2q \cdot k = k^2 + q^2 - (k-q)^2 \tag{A.11}$$

to write the left-hand sides of (A.10) as the sum of rank-2 Feynman integrals which, with the exception of the one multiplied by $q^2$ from (A.11), have one less denominator factor. This formally determines the coefficients $a^{\mu\nu}$ and $b^{\mu\nu}$ in (A.8) in terms of a set of rank-2 integrations. The vector function $c^\mu$ is then found from the contraction

$$J_2^{\mu\nu}{}_{\nu} = p_{2,\nu} a^{\mu\nu} + q_\nu b^{\mu\nu} + e^\mu \tag{A.12}$$

This contraction eliminates the $k^2$ denominator term in the integrand of $J_2^{\mu\nu\lambda}$ in (A.7) and produces a vector-valued integral. Solving the system of algebraic equations (A.10) and (A.12) then formally determines the rank-3 Feynman integral (A.7) in terms of rank-1 and rank-2 Feynman integrals. The rank-2 Feynman integrals thus generated can then be evaluated in the same way by writing a decomposition for them analogous to (A.8) and solving for them in terms of vector- and scalar-valued Feynman integrals. Finally, the rank-1 integrations can be solved for in terms of a set of scalar-valued integrals, most of which have fewer denominator factors in their integrands.

After reducing all tensor Feynman integrals to scalar ones in the ladder amplitudes (4.57)–(4.59), the resulting collection of scalar integrals can be decomposed further using (A.5). All of these scalar integrations can then be evaluated using the Feynman parametrizations

$$\begin{aligned} \frac{1}{a^n b} &= n \int_0^1 dx \, \frac{x^{n-1}}{[(1-x)b + xa]^{n+1}} \\ \frac{1}{a^n bc} &= n(n+1) \int_0^1 dx \int_0^x dy \, \frac{y^{n-1}}{[ay + b(x-y) + c(1-x)]^{n+2}} \end{aligned} \tag{A.13}$$

and the (2 + 1)-dimensional Feynman integral identity

$$\int \frac{d^3k}{(2\pi)^3} \, \frac{1}{(k^2 + 2k \cdot p + \alpha)^r} = \frac{\Gamma(r - 3/2)}{8\pi^{3/2}\Gamma(r)} \frac{1}{(\alpha - p^2)^{r-3/2}} \tag{A.14}$$

which are valid for $n \geq 1$ and $2r > 3$. In (A.13) the denominator factors $a$, $b$ and $c$ will be various combinations of the meson, photon and graviton propagator denominator factors which appear in (4.57)–(4.59). The above techniques reduce all Feynman integrations to scalar-valued ones which require no more than three Feynman parameters for their explicit evaluation. Some of the Feynman parameter integrations arising from the three denominator parametrization in (A.13) turn out to be quite complicated and not explicitly computable because of the required double integration. They can be simplified, however, in the infrared limit where $M, \mu \to \infty$. After all of these integrations are carried out, the individual integrals can be expanded about $q^2 = 0$. Very few of these expansions will contribute a singular simple pole term at $q^2 = 0$. When combined into the expressions for the ladder amplitudes, we arrive at the leading singular contributions in (4.62). For more details about the above methods and identities, see [30].

# Appendix B Feynman Integrals in the Infrared Limit

In this Appendix we simply list the results of the Feynman integrations which contribute non-vanishing terms to the amplitudes (4.57)–(4.59) in the infrared limit. The remaining integrals that appear were either found to vanish in this limit or can be obtained from those listed below using the methods discussed in Appendix A and by permutations of the external meson momenta. These were all evaluated using the above techniques, and some additional results can be found in [30]. As discussed before, many of the Feynman integrals that one encounters vanish in this regime of the quantum field theory. Below we have only listed the distinct integrals which produce non-vanishing results in this limit, i.e. those with singular pieces at $q^2 = 0$. We have also listed the results for the parameter values $M = \mu$, since, as mentioned before, the conformal dimensions obtained from these parts of the amplitudes will be independent of all masses in the theory.

$$\int \frac{d^3k}{(2\pi)^3} \, \frac{k^\mu}{D_g(0,\mu)D_g(q,0)D_g(q,M)D_m(p_2)} = -\frac{\pi i}{8M^3 q^2} \, q^\mu$$

$$\begin{aligned}
&\int \frac{d^3k}{(2\pi)^3} \, \frac{k^\mu k^\nu}{D_g(0,\mu)D_g(q,0)D_g(q,M)D_m(p_2)} \\
&= \frac{\pi i}{16Mq^2} \left\{ \eta^{\mu\nu} + \frac{1}{2m^2} \left( \pi^{-2} + 2(p_2 \cdot q)/q^2 \right) (p_2^\mu q^\nu + q^\mu p_2^\nu) + \frac{2}{q^2} q^\mu q^\nu - \frac{1}{m^2} p_2^\mu p_2^\nu \right\}
\end{aligned}$$

$$\begin{aligned}
&\int \frac{d^3k}{(2\pi)^3} \, \frac{k^\mu k^\nu}{k^2 D_g(q,0)D_g(q,M)D_m(p_2)} \\
&= \frac{\pi i}{16Mq^2} \left\{ \left(1 - 2q^2(1+\pi^{-2})/(p_2 \cdot q)\right) \eta^{\mu\nu} - \frac{3}{2} \left(1 + \pi^{-2}\right) (p_2^\mu q^\nu + q^\mu p_2^\nu) \right. \\
&\qquad \left. + \frac{2}{p_2 \cdot q} \left(1 + \pi^{-2}\right) q^\mu q^\nu + \frac{q^2}{m^2 (p_2 \cdot q)} \left(4(1+\pi^{-2}) + (p_2 \cdot q)/q^2\right) p_2^\mu p_2^\nu \right\}
\end{aligned}$$

$$\begin{aligned}
&\int \frac{d^3k}{(2\pi)^3} \, \frac{k^\mu k^\nu}{k^2 D_g(0,\mu)D_g(q,0)D_g(q,M)D_m(p_2)} \\
&= \frac{\pi i}{8M^3 (p_2 \cdot q)} \left\{ (1+\pi^{-2})\eta^{\mu\nu} + \frac{p_2 \cdot q}{2m^2 q^2} \left(3 + 4\pi^{-2} + 2(p_2 \cdot q)/q^2\right) (p_2^\mu q^\nu + q^\mu p_2^\nu) \right. \\
&\qquad \left. - \frac{1}{q^2} \left(1 + \pi^{-2} + (p_2 \cdot q)/q^2\right) q^\mu q^\nu - \frac{1}{m^2} \left(3 + 2\pi^{-2}\right) p_2^\mu p_2^\nu \right\}
\end{aligned}$$

$$J_2^{\mu\nu\lambda} = -\frac{\pi i}{32Mm^2q^2}\ \eta^{\mu\nu}p_2^\lambda$$

$$\begin{aligned} J_1^{\mu\nu\lambda} \quad &= \quad \frac{\pi i}{16M^3q^2}\left\{\frac{1+\pi^{-2}}{p_2\cdot q}\left(1+\frac{p_2\cdot q}{q^2+p_2\cdot q}\right)\eta^{\mu\nu}q^\lambda\right. \\ &\qquad \left. -\frac{(1+\pi^{-2})}{m^2}\left(1+\frac{q^2}{p_2\cdot q}+\frac{p_2\cdot q}{q^2+p_2\cdot q}\right)\eta^{\mu\nu}p_2^\lambda\right\} \end{aligned}$$

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